



Chapter 2

The Projective Class Group

2.1 Introduction

This chapter is devoted to the *projective class group* $K_0(R)$ of a ring R .

We give in Section 2.2 three equivalent definitions of $K_0(R)$, namely, by the universal additive invariant for finitely generated projective modules, by the Grothendieck construction applied to the abelian monoid of isomorphism classes of finitely generated projective modules, and by idempotent matrices, and discuss the significance of $K_0(R)$ for the category of finitely generated projective modules. Some calculations for principal ideal domains and Dedekind rings are provided in Section 2.3.

We explain the connections to geometry. We prove *Swan's Theorem 2.27*, which identifies $K_0(C^0(X))$ for the ring $C^0(X)$ of continuous functions on a compact space X with the Grothendieck group of the abelian monoid of isomorphism classes of vector bundles over X , see (2.31). The relevance of $K_0(\mathbb{Z}G)$ for topologists is illustrated by *Wall's finiteness obstruction*, which also leads to a geometric description of $K_0(\mathbb{Z}G)$ in terms of finitely dominated spaces and is discussed in detail in Section 2.5.

We introduce variants of the *K-theoretic Farrell-Jones Conjecture* for projective class groups in Section 2.8. A prototype asserts that for a torsionfree group G and a regular ring R , e.g., $R = \mathbb{Z}$ or R a field, the change of rings map

$$K_0(R) \xrightarrow{\cong} K_0(RG)$$

is bijective. It implies the conjecture that for a torsionfree group G the reduced projective class group $\tilde{K}_0(\mathbb{Z}G)$ vanishes, which is for finitely presented G equivalent to the conjecture that every finitely dominated CW -complex with $\pi_1(X) \cong G$ is homotopy equivalent to a finite CW -complex. We also introduce a version where the group is not necessarily torsionfree, but R is a regular ring with $\mathbb{Q} \subseteq R$ or a field of prime characteristic.

In Section 2.9 we consider *Kaplansky's Idempotent Conjecture*, which asserts for a torsionfree group G and a field F that 0 and 1 are the only idempotents in FG . It is a consequence of the Farrell-Jones Conjecture. We also discuss various *Bass Conjectures*, all of which are implied by the Farrell-Jones Conjecture, in Section 2.10.

Finally, we give a survey of $K_0(\mathbb{Z}G)$ for finite groups G and of $K_0(C_r^*(G))$ in Section 2.12 and of $K_0(\mathcal{N}(G))$ in Section 2.13, where $C_r^*(G)$ is the reduced group C^* -algebra and $\mathcal{N}(G)$ the group von Neumann algebra.

2.2 Definition and Basic Properties of the Projective Class Group

Definition 2.1 (Projective class group $K_0(R)$). Let R be an (associative) ring (with unit). Define its *projective class group* $K_0(R)$ to be the abelian group whose generators are isomorphism classes $[P]$ of finitely generated projective R -modules P and whose relations are $[P_0] + [P_2] = [P_1]$ for any exact sequence $0 \rightarrow P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow 0$ of finitely generated projective R -modules.

Define $G_0(R)$ analogously but replacing finitely generated projective by finitely generated.

Given a ring homomorphism $f: R \rightarrow S$, we can assign to an R -module M an S -module f_*M by $S \otimes_R M$ where we consider S as a right R -module using f . We say that f_*M is obtained by *induction with f* from M . If M is finitely generated or free or projective, the same is true for f_*M . This construction is natural, compatible with direct sums, and sends an exact sequence $0 \rightarrow P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow 0$ of finitely generated projective R -modules to an exact sequence $0 \rightarrow f_*P_0 \rightarrow f_*P_1 \rightarrow f_*P_2 \rightarrow 0$ of finitely generated projective S -modules. Hence we get a homomorphism of abelian groups

$$(2.2) \quad f_* = K_0(f): K_0(R) \rightarrow K_0(S), \quad [P] \mapsto [f_*P],$$

which is also called the *change of rings homomorphism*. Thus K_0 becomes a covariant functor from the category of rings to the category of abelian groups.

Remark 2.3 (The universal property of the projective class group). One should view $K_0(R)$ together with the assignment sending a finitely generated projective R -module P to its class $[P]$ in $K_0(R)$ as the *universal additive invariant* or the *universal dimension function* for finitely generated projective R -modules. Namely, suppose that we are given an abelian group and an assignment d that associates to a finitely generated projective R -module an element $d(P) \in A$ such that $d(P_0) + d(P_2) = d(P_1)$ holds for any exact sequence $0 \rightarrow P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow 0$ of finitely generated projective R -modules. Then there is precisely one homomorphism of abelian groups $\phi: K_0(R) \rightarrow A$ such that $\phi([P]) = d(P)$ holds for every finitely generated projective R -module P . The analogous statement holds for $G_0(R)$ if we consider finitely generated R -modules instead of finitely generated projective R -modules.

A ring is an *integral domain* if every zero-divisor is trivial, i.e., if $r, s \in R$ satisfy $rs = 0$, then $r = 0$ or $s = 0$. A *principal ideal domain* is a commutative integral domain for which every ideal is a *principal ideal*, i.e., of the form $(r) = \{r'r \mid r' \in R\}$ for some $r \in R$.

Example 2.4 ($K_0(R)$ and $G_0(R)$ of a principal ideal domain). Let R be a principal ideal domain. Then we get isomorphisms of abelian groups

$$\begin{aligned} \mathbb{Z} &\xrightarrow{\cong} K_0(R), \quad n \mapsto [R^n]; \\ K_0(R) &\xrightarrow{\cong} G_0(R), \quad [P] \mapsto [P]. \end{aligned}$$

This follows from the structure theorem of finitely generated R -modules over principal ideal domains. It implies that any finitely generated R -module M can be written as a direct sum $R^n \oplus T$ for some torsion R -module T for which there exists an exact sequence of R -modules of the shape $0 \rightarrow R^s \rightarrow R^s \rightarrow T \rightarrow 0$. Moreover, M is projective if and only if T is trivial and we have $R^m = R^n \iff m = n$.

Definition 2.5 (Reduced projective class group $K_0(R)$). Define the *reduced projective class group* $\tilde{K}_0(R)$ to be the quotient of $K_0(R)$ by the abelian subgroup $\{[R^m] - [R^n] \mid n, m \in \mathbb{Z}, m, n \geq 0\}$, which is the same as the abelian subgroup generated by the class $[R]$.

We conclude from Example 2.4 that the reduced projective class group $\tilde{K}_0(R)$ is isomorphic to the cokernel of the homomorphism

$$f_*: K_0(\mathbb{Z}) \rightarrow K_0(R)$$

where f is the unique ring homomorphism $\mathbb{Z} \rightarrow R$, $n \mapsto n \cdot 1_R$.

Remark 2.6 (The projective class group as a Grothendieck group). Let $\text{Proj}(R)$ be the abelian semigroup of isomorphism classes of finitely generated projective R -modules with the addition coming from the direct sum. Let $K'_0(R)$ be the associated abelian group given by the Grothendieck construction applied to $\text{Proj}(R)$. There is a natural homomorphism

$$\phi: K'_0(R) \xrightarrow{\cong} K_0(R)$$

sending the class of a finitely generated projective R -module P in $K'_0(R)$ to its class in $K_0(R)$. This is a well-defined isomorphism of abelian groups.

The analogous definition of $G'_0(R)$ and the construction of a homomorphism $G'_0(R) \rightarrow G_0(R)$ makes sense, but the latter map is *not* bijective in general. It works for $K_0(R)$ because every exact sequence of projective R -modules $0 \rightarrow P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow 0$ splits and thus yields an isomorphism $P_1 \cong P_0 \oplus P_2$. In general K -theory deals with exact sequences, not with direct sums. Therefore Definition 2.1 of $K_0(R)$ reflects better the underlying idea of K -theory than its definition in terms of the Grothendieck construction.

Exercise 2.7. Prove that the homomorphism $\phi: K'_0(R) \rightarrow K_0(R)$ appearing in Remark 2.6 is a well-defined isomorphism of abelian groups.

Remark 2.8 (What does the reduced projective class group measure?). Let P be a finitely generated projective R -module. Then we conclude from Remark 2.6 that its class $[P] \in \tilde{K}_0(R)$ is trivial if and only if P is *stably finitely generated free*, i.e., $P \oplus R^r \cong R^s$ for appropriate integers $r, s \geq 0$. So the reduced projective class group $\tilde{K}_0(R)$ measures the deviation of a finitely generated projective R -module from being stably finitely generated free. Note that, in general, stably finitely generated free does not imply finitely generated free, as Examples 2.9 and 2.28 will show.

Example 2.9 (Dunwoody's example). An interesting $\mathbb{Z}G$ -module P that is stably finitely generated free but not finitely generated free is constructed by Dunwoody [317] for G the torsionfree one-relator group $\langle a, b \mid a^2 = b^3 \rangle$, which is the

fundamental group of the trefoil knot. Note that $\widetilde{K}_0(\mathbb{Z}G)$ is known to be trivial, in other words, every finitely generated projective RG -module is stably finitely generated free. It is also worth mentioning that $\mathbb{Z}G$ contains no idempotent besides 0 and 1. Hence any direct summand in $\mathbb{Z}G$ is free.

More examples of this kind are given in Berridge-Dunwoody [134].

One basic feature of algebraic K -theory is *Morita equivalence*.

Theorem 2.10 (Morita equivalence for $K_0(R)$). *For every ring R and integer $n \geq 1$, there is a natural isomorphism*

$$\mu: K_0(R) \xrightarrow{\cong} K_0(M_n(R)).$$

Proof. We can consider R^n as an $M_n(R)$ - R -bimodule, denoted by ${}_{M_n(R)}R^n_R$. Then μ sends $[P]$ to $[{}_{M_n(R)}R^n_R \otimes_R P]$. We can also consider R^n as an R - $M_n(R)$ -bimodule denoted by ${}_R R^n_{M_n(R)}$. Define $\nu: K_0(M_n(R)) \rightarrow K_0(R)$ by sending $[Q]$ to $[{}_R R^n_{M_n(R)} \otimes_{M_n(R)} Q]$. Then μ and ν are inverse to one another. $\square$

Exercise 2.11. Check that μ and ν are inverse to one another.

We omit the easy proof of the next lemma.

Lemma 2.12. *Let R_0 and R_1 be rings. Denote by $\text{pr}_i: R_0 \times R_1 \rightarrow R_i$ for $i = 0, 1$ the projection. Then we obtain an isomorphism*

$$(\text{pr}_0)_* \times (\text{pr}_1)_*: K_0(R_0 \times R_1) \xrightarrow{\cong} K_0(R_0) \times K_0(R_1).$$

Example 2.13 (Rings with non-trivial $\widetilde{K}_0(R)$). We conclude from Example 2.4 and Lemma 2.12 that for a principal ideal domain R we have

$$\begin{aligned} K_0(R \times R) &\cong \mathbb{Z} \oplus \mathbb{Z}; \\ \widetilde{K}_0(R \times R) &\cong \mathbb{Z}. \end{aligned}$$

The $R \times R$ -module $R \times \{0\}$ is finitely generated projective but not stably finitely generated free. It is a generator of the infinite cyclic group $\widetilde{K}_0(R \times R)$.

Notation 2.14 ($M(R)$, $\text{GL}(R)$, and $\text{Idem}(R)$). Let $M_{m,n}(R)$ be the set of (m, n) -matrices over R . For $A \in M_{m,n}(R)$, let $r_A: R^m \rightarrow R^n$, $x \rightarrow xA$ be the R -homomorphism of (left) R -modules given by right multiplication by A . Let $M_n(R)$ be the ring of (n, n) -matrices over R . Denote by $\text{GL}_n(R)$ the group of invertible (n, n) -matrices over R . Let $\text{Idem}_n(R)$ be the subset of $M_n(R)$ of idempotent matrices A , i.e., (n, n) -matrices satisfying $A^2 = A$. There are embeddings $i_{t,n}: M_n(R) \rightarrow M_{n+1}(R)$, $A \mapsto \begin{pmatrix} A & 0 \\ 0 & t \end{pmatrix}$ for $t = 0, 1$ and $n \geq 1$. The embedding $i_{1,n}$ induces an embedding $\text{GL}_n(R) \rightarrow \text{GL}_{n+1}(R)$ of groups. Let $\text{GL}(R)$ be the union of the $\text{GL}_n(R)$ -s, which is a group again. Denote by $M(R)$ the union of the $M_n(R)$ -s with respect to the embeddings i_0 . This is a ring without unit. Let $\text{Idem}(R)$ be the set of idempotent elements in $M(R)$. This is the same as the union of the

$\text{Idem}_n(R)$ -s with respect to the embeddings $\text{Idem}_n(R) \rightarrow \text{Idem}_{n+1}(R)$ coming from the embeddings $i_{0,n}: M_n(R) \rightarrow M_{n+1}(R)$.

Remark 2.15 (The projective class groups in terms of idempotent matrices). The projective class groups $K_0(R)$ can also be defined in terms of idempotent matrices. Namely, the conjugation action of $\text{GL}_n(R)$ on $M_n(R)$ induces an action of $\text{GL}(R)$ on $M(R)$ which leaves $\text{Idem}(R)$ fixed. One obtains a bijection of sets

$$\phi: \text{GL}(R) \backslash \text{Idem}(R) \rightarrow \text{Proj}(R), \quad [A] \mapsto \text{im}(r_A: R^n \rightarrow R^n).$$

This becomes a bijection of abelian semigroups if we equip the source with the addition coming from $(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and the target with the one coming from the direct sum. So we can identify $K_0(R)$ with the Grothendieck group associated to the abelian semigroup $\text{GL}(R) \backslash \text{Idem}(R)$ by Remark 2.6.

Exercise 2.16. Show that the map ϕ appearing in Remark 2.15 is a well-defined isomorphism of abelian semigroups.

Example 2.17 (A ring R with trivial $K_0(R)$). Let F be a field and let V be an F -vector space with an infinite countable basis. Consider the ring $R = \text{end}_F(V)$. Next we prove that $K_0(R)$ is trivial.

By Remark 2.15 it suffices to show for every integer $n \geq 0$ and two idempotent matrices $A, B \in \text{Idem}_n(R)$ that the matrices $A \oplus 0 \oplus 1$ and $B \oplus 0 \oplus 1$ in $M_{n+2}(R)$ are conjugate by an element in $\text{GL}_{n+2}(R)$. This follows from the observations that both the kernel and the image of the F -linear endomorphisms $r_{A \oplus 0 \oplus 1}$ and $r_{B \oplus 0 \oplus 1}$ of V^{n+2} have infinite countable dimension, two F -vector spaces of infinite countable dimension are isomorphic, and the inclusions induce isomorphisms $\ker(r_{A \oplus 0 \oplus 1}) \oplus \text{im}(r_{A \oplus 0 \oplus 1}) \xrightarrow{\cong} V^{n+2}$, and $\ker(r_{B \oplus 0 \oplus 1}) \oplus \text{im}(r_{B \oplus 0 \oplus 1}) \xrightarrow{\cong} V^{n+2}$.

Lemma 2.18. Let G be a group. Let R be a commutative integral domain with quotient field F . Then we obtain an isomorphism

$$K_0(RG) \xrightarrow{\cong} \widetilde{K}_0(RG) \oplus \mathbb{Z}, \quad [P] \mapsto ([P], \dim_F(F \otimes_{RG} P))$$

where F is considered as an RG -module with respect to the trivial G -action and the inclusion of rings $j: R \rightarrow F$.

Proof. Since $F \otimes_{RG} P$ is a finite-dimensional F -vector space for finitely generated P and $F \otimes_{RG} (P \oplus Q) \cong_G (F \otimes_{RG} P) \oplus (F \otimes_{RG} Q)$, this is a well-defined homomorphism. Bijectivity follows from $\dim_F(F \otimes_{RG} RG^n) = n$. $\square$

2.3 The Projective Class Group of a Dedekind Domain

Let R be a commutative integral domain with quotient field F . A non-zero R -submodule $I \subset F$ is called a *fractional ideal* if for some $r \in R$ we have $rI \subseteq R$. A fractional ideal I is called *principal* if I is of the form $\{\frac{ra}{b} \mid r \in R\}$ for some $a, b \in R$ with $a, b \neq 0$.

Definition 2.19 (Dedekind domain). A commutative integral domain R is called a *Dedekind ring* if for any fractional ideal I there exists another fractional ideal J with $IJ = R$.

Note that in Definition 2.19 the fractional ideal J must be given by $\{x \in F \mid x \cdot I \subseteq R\}$.

The fractional ideals in a Dedekind ring form by definition a group under multiplication of ideals with R as unit. The principal fractional ideals form a subgroup. The *class group* $C(R)$ is the quotient of these abelian groups.

A proof of the next theorem can be found for instance in [727, Corollary 11 on page 14] and [860, Theorem 1.4.12 on page 20].

Theorem 2.20 (The reduced projective class group and the class group of Dedekind domains). *Let R be a Dedekind domain. Then every fractional ideal is a finitely generated projective R -module and we obtain an isomorphism of abelian groups*

$$\mathbb{Z} \oplus C(R) \xrightarrow{\cong} K_0(R), \quad (n, [I]) \mapsto n \cdot [R] + [I] - [R].$$

In particular, we get an isomorphism

$$C(R) \xrightarrow{\cong} \widetilde{K}_0(R), \quad [I] \mapsto [I].$$

A ring is called *hereditary* if every ideal is projective, or, equivalently, if every submodule of a projective R -module is projective, see [215, Theorem 5.4 in Chapter I.5 on page 14].

Theorem 2.21 (Characterization of Dedekind domains). *The following assertions are equivalent for a commutative integral domain with quotient field F :*

- (i) R is a Dedekind domain;
- (ii) For every pair of ideals $I \subseteq J$ of R , there exists an ideal $K \subseteq R$ with $I = JK$;
- (iii) R is hereditary;
- (iv) Every finitely generated torsionfree R -module is projective;
- (v) R is Noetherian and integrally closed in its quotient field F and every non-zero prime ideal is maximal.

Proof. This follows from [271, Proposition 4.3 on page 76 and Proposition 4.6 on page 77] and the fact that a finitely generated torsionfree module over an integral domain R can be embedded into R^n for some integer $n \geq 0$. See also [57, Chapter 13]. $\square$

Remark 2.22 (The class group in terms of ideals of R). One calls two ideals I and J in R equivalent if there exist non-zero elements r and s in R with $rI = sJ$. Then $C(R)$ is the same as the equivalence classes of ideals under multiplication of ideals and the class given by the principal ideals as unit. Two ideals I and J of R define the same element in $C(R)$ if and only if they are isomorphic as R -modules, see [860, Proposition 1.4.4 on page 17].

Recall that an *algebraic number field* is a finite algebraic extension of $\mathbb{Q}$ and the *ring of integers* in F is the integral closure of $\mathbb{Z}$ in F .

Theorem 2.23 (The class group of a ring of integers is finite). *Let R be the ring of integers in an algebraic number field. Then R is a Dedekind domain and its class group $C(R)$ and hence its reduced projective class group $\tilde{K}_0(R)$ are finite.*

Proof. See [860, Theorem 1.4.18 on page 22 and Theorem 1.4.19 on page 23]. $\square$

Remark 2.24 (Class group of $\mathbb{Z}[\exp(2\pi i/p)]$). Let p be a prime number. The ring of integers in the algebraic number field $\mathbb{Q}[\exp(2\pi i/p)]$ is $\mathbb{Z}[\exp(2\pi i/p)]$. Its class group $C(\mathbb{Z}[\exp(2\pi i/p)])$ is finite by Theorem 2.23. However, its structure as a finite abelian group is only known for finitely many small primes, see [727, Remark 3.4 on page 30] or [990, Tables §3 on page 352ff].

Example 2.25 ($\tilde{K}_0(\mathbb{Z}[\sqrt{-5}])$). The reduced projective class group $\tilde{K}_0(\mathbb{Z}[\sqrt{-5}])$ of the Dedekind domain $\mathbb{Z}[\sqrt{-5}]$ is cyclic of order two. A generator is given by the maximal ideal $(3, 2 + \sqrt{-5})$ in $\mathbb{Z}[\sqrt{-5}]$. (For more details see [860, Exercise 1.4.20 on page 25]).

2.4 Swan's Theorem

Let F be the field $\mathbb{R}$ or $\mathbb{C}$. Let X be a compact space. Denote by $C(X, F)$ or briefly by $C(X)$ the ring of continuous functions from X to F . Let ξ and η be (finite-dimensional locally trivial) F -vector bundles over X . Denote by $C(\xi)$ the F -vector space of continuous sections of ξ . This becomes a $C(X)$ -module under pointwise multiplication. If $\underline{F}$ denotes the trivial 1-dimensional vector bundle $X \times F \rightarrow X$, then $C(\underline{F})$ and $C(X)$ are isomorphic as $C(X)$ -modules. If ξ and η are isomorphic as F -vector bundles, then $C(\xi)$ and $C(\eta)$ are isomorphic as $C(X)$ -modules. There is an obvious isomorphism of $C(X)$ -modules

$$(2.26) \quad C(\xi) \oplus C(\eta) \xrightarrow{\cong} C(\xi \oplus \eta).$$

Since X is compact, every F -vector bundle has a finite bundle atlas and admits a Riemannian metric. This implies the existence of an F -vector bundle ξ' such that $\xi \oplus \xi'$ is isomorphic as an F -vector bundle to a trivial F -vector bundle $\underline{F}^n$. Hence $C(\xi)$ is a finitely generated projective $C(X)$ -module. Denote by $\text{hom}(\xi, \eta)$ the $C(X)$ -module of morphisms of F -vector bundles from ξ to η , i.e., of continuous

maps between the total spaces that commutes with the bundle projections to X and induce linear (not necessarily injective or bijective) maps between the fibers over x for all $x \in X$. This becomes a $C(X)$ -module under pointwise multiplication. Such a morphism $f: \xi \rightarrow \eta$ induces a $C(X)$ -homomorphism $C(f): C(\xi) \rightarrow C(\eta)$ by composition. The next result is due to Swan [939].

Theorem 2.27 (Swan's Theorem). *Let X be a compact space and $F = \mathbb{R}, \mathbb{C}$. Then:*

- (i) *Let ξ and η be F -vector bundles. Then we obtain an isomorphism of $C(X)$ -modules*

$$\Gamma(\xi, \eta): \text{hom}(\xi, \eta) \rightarrow \text{hom}_{C(X)}(C(\xi), C(\eta)), \quad f \mapsto C(f);$$

- (ii) *We have $\xi \cong \eta \iff C(\xi) \cong_{C(X)} C(\eta)$;*

- (iii) *If P is a finitely generated projective $C(X)$ -module, then there exists an F -vector bundle ξ satisfying $C(\xi) \cong_{C(X)} P$.*

Proof. (i) Obviously $\Gamma(\xi \oplus \xi', \eta)$ can be identified with $\Gamma(\xi, \eta) \oplus \Gamma(\xi', \eta)$ and $\Gamma(\xi, \eta \oplus \eta')$ can be identified with $\Gamma(\xi, \eta) \oplus \Gamma(\xi, \eta')$ under the identification (2.26). Since a direct sum of two maps is a bijection if and only if each of the maps is a bijection and for every ξ there is an ξ' such that $\xi \oplus \xi'$ is trivial, it suffices to treat the case where $\xi = \underline{F}^m$ and $\eta = \underline{F}^n$ for appropriate integers $m, n \geq 0$. There is an obvious commutative diagram

$$\begin{array}{ccc} \text{hom}(\underline{F}^m, \underline{F}^n) & \xrightarrow{\Gamma(\underline{F}^m, \underline{F}^n)} & \text{hom}_{C(X)}(C(\underline{F}^m), C(\underline{F}^n)) \\ \cong \downarrow & & \downarrow \cong \\ M_{m,n}(\text{hom}(\underline{F}, \underline{F})) & \xrightarrow{M_{m,n}(\Gamma(\underline{F}, \underline{F}))} & M_{m,n}(C(\underline{F})). \end{array}$$

Hence it suffices to treat the claim for $m = n = 1$, which is obvious.

- (ii) This follows from assertion (i).

- (iii) Given a finitely generated projective $C(X)$ -module P , choose a $C(X)$ -map $p: C(X)^n \rightarrow C(X)^n$ satisfying $p^2 = p$ and $\text{im}(p) \cong_{C(X)} P$. Because of assertion (ii) we can choose a morphism of F -vector bundles $q: \underline{F}^n \rightarrow \underline{F}^n$ with $\Gamma(\underline{F}^n, \underline{F}^n)(q) = p$. We conclude $q^2 = q$ from $p^2 = p$ and the injectivity of $\Gamma(\underline{F}^n, \underline{F}^n)$. Elementary bundle theory shows that the image of q and the image of $1 - q$ are F -subvector bundles in $\underline{F}^n$ satisfying $\text{im}(q) \oplus \text{im}(1 - q) = \underline{F}^n$. One easily checks $C(\text{im}(q)) \cong_{C(X)} P$. $\square$

One may summarize Theorem 2.27 by saying that we obtain an equivalence of $C(X)$ -additive categories from the category of F -vector bundles over X to the category of finitely generated projective $C(X)$ -modules by sending ξ to $C(\xi)$.

Example 2.28 ($C(TS^n)$). Consider the n -dimensional sphere S^n . Let TS^n be its tangent bundle. Then $C(TS^n)$ is a finitely generated projective $C(S^n)$ -module. It is free if and only if TS^n is trivial. This is equivalent to the condition that $n = 1, 3, 7$, see [155]. On the other hand $C(TS^n)$ is always stably finitely generated free as a $C(S^n)$ -module, since TS^n is stably finitely generated free as an F -vector bundle because the direct sum of TS^n and the normal bundle $\nu(S^n, \mathbb{R}^{n+1})$ of the standard embedding $S^n \subseteq \mathbb{R}^{n+1}$ is $T\mathbb{R}^{n+1}|_{S^n}$ and both F -vector bundles $\nu(S^n, \mathbb{R}^{n+1})$ and $T\mathbb{R}^{n+1}|_{S^n}$ are trivial.

Exercise 2.29. Consider an integer $n \geq 1$. Show that there exists a $C(S^n)$ -module M with $C(TS^n) \cong_{C(S^n)} C(S^n) \oplus M$ if and only if S^n admits a nowhere vanishing vector field. (This is equivalent to requiring that $\chi(S^n) = 0$, or, equivalently, that n is odd.)

Remark 2.30 (Topological K -theory in dimension 0). Let X be a compact space. Let $\text{Vect}_F(X)$ be the abelian semigroup of isomorphism classes of F -vector bundles over X where the addition comes from the Whitney sum. Let $K^0(X)$ be the abelian group obtained from the Grothendieck construction to it. It is called the 0-th topological K -group of X . If $f: X \rightarrow Y$ is a map of compact spaces, the pullback construction yields a homomorphism $K^0(f): K^0(Y) \rightarrow K^0(X)$. Thus we obtain a contravariant functor K^0 from the category of compact spaces to the category of abelian groups. Since the pullback of a vector bundle with two homotopic maps yields isomorphic vector bundles, $K^0(f)$ depends only on the homotopy class of f . Actually there is a sequence of such homotopy invariant covariant functors K^n for $n \in \mathbb{Z}$ that constitutes a generalized cohomology theory K^* called *topological K -theory*. It is 2-periodic if $F = \mathbb{C}$, i.e., there are natural so-called *Bott isomorphisms* $K^n(X) \xrightarrow{\cong} K^{n+2}(X)$ for $n \in \mathbb{Z}$. If $F = \mathbb{R}$, it is 8-periodic. We will give further explanations and generalizations of topological K -theory later in Section 10.2

Swan's Theorem 2.27 yields an identification

$$(2.31) \quad K^0(X) \cong K_0(C(X)) \quad [\xi] \mapsto [C^0(\xi)].$$

Exercise 2.32. Let $f: X \rightarrow Y$ be a map of compact spaces. Composition with f yields a ring homomorphism $C(f): C(Y) \rightarrow C(X)$. Show that under the identification (2.31) the maps $K^0(f): K^0(Y) \rightarrow K^0(X)$ and $C(f)_*: K_0(C(Y)) \rightarrow K_0(C(X))$ coincide.

Exercise 2.33. Compute $K_0(C(D^n))$ for the n -dimensional disk D^n for $n \geq 0$.

2.5 Wall's Finiteness Obstruction

We now discuss the geometric relevance of $\tilde{K}_0(\mathbb{Z}G)$.

Let X be a CW -complex. It is called *finite* if it consists of finitely many cells. This is equivalent to the condition that X is compact. We call X *finitely dominated* if there exists a *finite domination* (Y, i, r) , i.e., a finite CW -complex Y together with

maps $i: X \rightarrow Y$ and $r: Y \rightarrow X$ such that $r \circ i$ is homotopic to the identity on X . If X is finitely dominated, its set of path components $\pi_0(X)$ is finite and the fundamental group $\pi_1(C)$ of each component C of X is finitely presented, see Lemma 2.42.

While studying existence problems for compact manifolds with prescribed properties (like for example the existence of certain group actions), it happens occasionally that it is relatively easy to construct a finitely dominated CW -complex with the desired property within a given homotopy type, whereas it is not at all clear whether one can also find a homotopy equivalent finite CW -complex. If the goal is to construct a compact manifold, this is a necessary step in the construction. Wall's finiteness obstruction, which we will explain below, decides this question.

An example of such a geometric problem is the *Spherical Space Form Problem* 9.205, i.e., the classification of closed manifolds M whose universal coverings are diffeomorphic or homeomorphic to the standard sphere. Such examples arise as unit spheres in unitary representations of finite groups, but there are also examples that do not occur in this way. This problem initiated not only the theory of the finiteness obstruction, but also surgery theory for closed manifolds with non-trivial fundamental group. We refer to the survey articles [284] and [694] for more information about the Spherical Space Form problem. It was finally solved by Madsen-Thomas-Wall [701, 702].

The finiteness obstruction also appears in the Ph.D.-thesis [915] of Siebenmann, who dealt with the problem whether a given smooth or topological manifold can be realized as the interior of a compact manifold with boundary.

Next we explain the definition and the main properties of the finiteness obstruction, illustrating that it is a kind of Euler characteristic, but now counting elements in the projective class group instead of counting ranks of finitely generated free modules.

2.5.1 Chain Complex Version of the Finiteness Obstruction

Definition 2.34 (Types of chain complexes). We call an R -chain complex *finitely generated*, *free*, or *projective* respectively if each R -chain module is finitely generated, free, or projective. It is called *positive* if $C_n = 0$ for $n \leq -1$. It is called *finite-dimensional* if there exists a natural number N such that $C_n = 0$ for $|n| \leq N$. It is called *finite* if it is finite-dimensional and finitely generated.

For the remainder of this section all chain complexes C_* are understood to be positive. Let R be a ring and C_* be an R -chain complex. A *finite domination* (F_*, i_*, p_*) of C_* consists of a finite free R -chain complex F_* and R -chain maps $i_*: C_* \rightarrow F_*$ and $r_*: F_* \rightarrow C_*$ such that $r_* \circ i_* \simeq \text{id}_{C_*}$ holds. The existence of a finite domination is equivalent to the existence of a finite projective R -chain complex P_* which is R -chain homotopy equivalence to C_* . For a proof of this claim we refer for instance to [644, Proposition 11.11 on page 222], or to the explicit construction in Subsection 23.7.5. For any such choice of P_* , define the *finiteness obstruction* $o(C_*) \in K_0(R)$ to be

$$(2.35) \quad o(C_*) := \sum_{n \geq 0} (-1)^n \cdot [P_n].$$

The *reduced finiteness obstruction* $\tilde{o}(C_*) \in \tilde{K}_0(R)$ is the image of $o(C_*)$ under the projection $K_0(R) \rightarrow \tilde{K}_0(R)$. The definition is indeed independent of the choice of P_* , since for two finite projective R -chain complexes P_* and Q_* coming with an R -chain homotopy equivalence $f_*: P_* \xrightarrow{\sim} Q_*$ the mapping cone $\text{cone}_*(f_*)$, see Definition 3.29, is contractible and hence we obtain an R -isomorphism

$$P_{\text{odd}} \oplus Q_{\text{ev}} \xrightarrow{\cong} P_{\text{ev}} \oplus Q_{\text{odd}}$$

from the isomorphism (3.30) and its inverse (3.31).

Lemma 2.36. (i) *If the two R -chain complexes C_* and D_* are R -chain homotopy equivalent and one of them is finitely dominated, then both are finitely dominated and we get*

$$o(C_*) = o(D_*);$$

(ii) *Let $0 \rightarrow C_* \rightarrow D_* \rightarrow E_* \rightarrow 0$ be an exact sequence of R -chain complexes. If two of the R -chain complexes C_* , D_* , and E_* are finitely dominated, then all three are finitely dominated and we get*

$$o(D_*) = o(C_*) + o(E_*);$$

(iii) *Let C_* be a finitely dominated R -chain complex. Then it is R -chain homotopy equivalent to a finite free R -chain complex if and only if $\tilde{o}(C_*)$ vanishes.*

Proof. (i) This follows directly from the definitions.

(ii) One can construct a commutative diagram of R -chain complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & C'_* & \longrightarrow & D'_* & \longrightarrow & E'_* \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & C_* & \longrightarrow & D_* & \longrightarrow & E_* \longrightarrow 0 \end{array}$$

such that the rows are exact, the upper row consists of finite projective R -chain complexes, and the vertical maps are R -chain homotopy equivalences, see for instance [644, Lemma 11.6 on page 216].

(iii) Suppose that $\tilde{o}(C_*) = 0$. Choose a finite projective R -chain complex P_* which is R -chain homotopy equivalent to C_* . An *elementary R -chain complex* E_* over an R -module M is an R -chain complex which is concentrated in two consecutive dimensions and its only non-trivial differential is given by $\text{id}_M: M \rightarrow M$. By adding elementary R -chain complexes over finitely generated free R -modules, one can arrange that P_* is of the shape $\cdots \rightarrow 0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0$ such that P_i is finitely generated free for $i \leq n-1$. Since $\tilde{o}(C_*) = (-1)^n \cdot [P_n] = 0$ holds in $\tilde{K}_0(R)$, the R -module P_n is stably free. Hence, by adding one further elementary

chain complex over a finitely generated free R -module, one can arrange that P_* is finite free. $\square$

2.5.2 Space Version of the Finiteness Obstruction

In the sequel we ignore base point questions. This is not a real problem since an inner automorphism of a group G induces the identity on $K_0(RG)$.

Given a finitely dominated connected CW -complex X with fundamental group π , we consider its universal covering $\tilde{X}$ and the associated cellular $\mathbb{Z}\pi$ -chain complex $C_*(\tilde{X})$. Given a finite domination (Y, i, r) , we regard the π -covering $\bar{Y}$ over Y associated to the epimorphism $r_*: \pi_1(Y) \rightarrow \pi_1(X)$. The pullback construction yields a π -covering $i^*\bar{Y}$ over X . Then $F_* = C_*(i^*\bar{Y})$ is a finite free $\mathbb{Z}\pi$ -chain complex. The maps i and r yield $\mathbb{Z}\pi$ -chain maps $r_*: F_* \rightarrow C_*(\tilde{X})$ and $i_*: C_*(\tilde{X}) \rightarrow F_*$ such that $r_* \circ i_*$ is $\mathbb{Z}\pi$ -chain homotopic to the identity on $C_*(\tilde{X})$. Thus (F_*, i_*, r_*) is a finite domination of the $\mathbb{Z}\pi$ -chain complex $C_*(\tilde{X})$. We have defined $o(C_*(\tilde{X})) \in K_0(\mathbb{Z}\pi)$ in (2.35). Now define the *unreduced finiteness obstruction*

$$(2.37) \quad o(X) := o(C_*(\tilde{X})) \in K_0(\mathbb{Z}\pi).$$

Define the *finiteness obstruction*

$$(2.38) \quad \tilde{o}(X) \in \tilde{K}_0(\mathbb{Z}\pi)$$

to be the image of $o(X)$ under the canonical projection $K_0(\mathbb{Z}\pi) \rightarrow \tilde{K}_0(\mathbb{Z}\pi)$. Obviously $\tilde{o}(X) = 0$ if X is homotopy equivalent to a finite CW -complex Z since in this case we can take $P_* = C_*(\tilde{Z})$ and $C_*(\tilde{Z})$ is a finite free $\mathbb{Z}\pi$ -chain complex. The next result is due to Wall, see [983] and [984].

Theorem 2.39 (Properties of the Finiteness Obstruction). *Let X be a finitely dominated connected CW -complex.*

- (i) *The space X is homotopy equivalent to a finite CW -complex if and only if $\tilde{o}(X)$ vanishes;*
- (ii) *Every element in $K_0(\mathbb{Z}G)$ can be realized as the finiteness obstruction $o(X)$ of a finitely dominated connected 3-dimensional CW -complex X with $G = \pi_1(X)$, provided that G is finitely presented.*

Theorem 2.39 illustrates why it is important to study the algebraic object $\tilde{K}_0(\mathbb{Z}\pi)$ when one is dealing with geometric or topological questions. The favorite case is when $\tilde{K}_0(\mathbb{Z}\pi)$ vanishes because then the finiteness obstruction is obviously zero and one does not have to make a specific computation of $\tilde{o}(X)$ in $\tilde{K}_0(\mathbb{Z}\pi)$.

Exercise 2.40. Let X be a finitely dominated connected CW -complex with fundamental group π . Define a homomorphism of abelian groups

$$\psi: K_0(\mathbb{Z}\pi) \rightarrow \mathbb{Z}, \quad [P] \mapsto \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}\pi} P).$$

Show that ψ sends $o(X)$ to the Euler characteristic $\chi(X)$.

Remark 2.41. One can extend the finiteness obstruction also to not necessarily connected CW-complexes. If X is a (not necessarily connected) finitely dominated CW-complex, we define

$$K_0(\mathbb{Z}[\pi_1(X)]) := \bigoplus_{C \in \pi_0(X)} K_0(\mathbb{Z}[\pi_1(C)]);$$

$$\tilde{K}_0(\mathbb{Z}[\pi_1(X)]) := \bigoplus_{C \in \pi_0(X)} \tilde{K}_0(\mathbb{Z}[\pi_1(C)]),$$

and the unreduced finite obstruction and the finiteness obstruction to be

$$o(X) := \{o(C) \mid C \in \pi_0(X)\} \in K_0(\mathbb{Z}[\pi_1(X)]);$$

$$\tilde{o}(X) := \{\tilde{o}(C) \mid C \in \pi_0(X)\} \in \tilde{K}_0(\mathbb{Z}[\pi_1(X)]).$$

Note that $K_0(\mathbb{Z}[\pi_1(X)])$ and $\tilde{K}_0(\mathbb{Z}[\pi_1(X)])$ are covariant functors in X in the obvious way.

For more information about the finiteness obstruction we refer for instance to [380, 382, 642, 669, 740, 743, 761, 838, 965, 983, 984].

2.5.3 Outline of the Proof of the Obstruction Property

In this subsection we outline the proof of Theorem 2.39.

The elementary proofs of the next two lemmas can be found in [983, Lemma 1.3] and [644, Lemma 14.8 on page 280].

Lemma 2.42. *Let G be a finitely presented group. Let $i: H \rightarrow G$ and $r: G \rightarrow H$ be group homomorphisms with $r \circ i = \text{id}_H$. Then H is finitely presented.*

Lemma 2.43. *Let G be a finitely generated group and H be a finitely presented group. Then the kernel $\ker(f)$ of any group epimorphism $f: G \rightarrow H$ is finitely generated as a normal subgroup, i.e., there exists a finite subset S of $\ker(f)$ such that the intersection of all normal subgroups of G containing S is $\ker(f)$.*

The next Lemma 2.44 follows from Lemma 2.42 and Lemma 2.43.

Lemma 2.44. *Let (Y, i, r) be a finite domination of the CW-complex X . Then we can arrange by attaching finitely many 2-cells to Y that the map $\pi_1(r): \pi_1(Y) \rightarrow \pi_1(X)$ is bijective and hence r is 2-connected.*

Lemma 2.45. *Let Y be a finitely dominated connected CW-complex whose finiteness obstruction $\tilde{o}(Y)$ vanishes. Then there are:*

- (i) A finite 2-dimensional connected CW-complex Z ;

- (ii) A 2-connected map $h: Z \rightarrow Y$;
- (iii) A finite free $\mathbb{Z}\pi$ -chain complex C_* with $C_*|_2 = C_*(\tilde{Z})$ and a $\mathbb{Z}\pi$ -chain homotopy equivalence $f_*: C_* \rightarrow C_*(\tilde{Y})$ with $f_*|_2 = C_*(\tilde{h})$, where here and in the sequel we identify $\pi = \pi_1(Z)$ with $\pi_1(Y)$ using the isomorphism $\pi_1(h): \pi_1(Z) \xrightarrow{\cong} \pi_1(Y)$.

Proof. By Lemma 2.44 we obtain a finite domination (Y, i, r) such that $r: Y \rightarrow X$ is 2-connected. Take Z to be the 2-skeleton Y_2 of Y and $h: Z \rightarrow X$ to be the restriction of r to Z .

Since h is 2-connected, the induced $\mathbb{Z}\pi$ -chain map $C_*(\tilde{h}): C_*(\tilde{Z}) \rightarrow C_*(\tilde{Y})$ is 2-connected and hence $H_n(\text{cone}_*(C_*(\tilde{h}))) = 0$ for $n \leq 2$. Let P_* be the $\mathbb{Z}\pi$ -subchain complex of $\text{cone}_*(C_*(\tilde{h}))$ given by

$$\dots \xrightarrow{c_5} \text{cone}_4(C_*(\tilde{h})) \xrightarrow{c_4} \text{cone}_3(C_*(\tilde{h})) \xrightarrow{c_3} \ker(c_2) \rightarrow 0 \rightarrow 0 \rightarrow 0$$

where c_* is the differential of $\text{cone}(C_*(\tilde{h}))$. Because of the exact sequence

$$0 \rightarrow \ker(c_2) \rightarrow \text{cone}_2(C_*(\tilde{h})) \xrightarrow{c_2} \text{cone}_1(C_*(\tilde{h})) \xrightarrow{c_1} \text{cone}_0(C_*(\tilde{h})) \rightarrow 0$$

the $\mathbb{Z}\pi$ -chain complex P_* is projective. The inclusion $i_*: P_* \rightarrow \text{cone}_*(C_*(\tilde{h}))$ is a homology equivalence of projective $\mathbb{Z}\pi$ -chain complexes and hence a $\mathbb{Z}\pi$ -chain homotopy equivalence. Put $Q_* = \Sigma^{-3}P_*$. Then Q_* is a positive projective $\mathbb{Z}\pi$ -chain complex such that $\Sigma^3 Q_*$ is $\mathbb{Z}\pi$ -chain homotopy equivalent to $\text{cone}_*(C_*(\tilde{h}))$.

The mapping cylinder $\text{cyl}(C_*(\tilde{h}))$, see Definition 3.29, is $\mathbb{Z}\pi$ -chain homotopy equivalent to $C_*(\tilde{Y})$ and there is an obvious short exact sequence of $\mathbb{Z}\pi$ -chain complexes

$$0 \rightarrow C_*(\tilde{Z}) \rightarrow \text{cyl}_*(C_*(\tilde{h})) \rightarrow \text{cone}(C_*(\tilde{h})) \rightarrow 0.$$

Since $C_*(\tilde{Z})$ is finite free and $C_*(\tilde{Y})$ is finitely dominated, we conclude from Lemma 2.36 (i) and (ii) that Q_* is finitely dominated and that we get in $\tilde{K}_0(\mathbb{Z}\pi)$

$$\begin{aligned} \tilde{o}(Q_*) &= -\tilde{o}(P_*) = -\tilde{o}(\text{cone}_*(C_*(\tilde{h}))) = \tilde{o}(\text{cyl}_*(C_*(\tilde{h}))) - \tilde{o}(C_*(\tilde{Z})) \\ &= \tilde{o}(C_*(\tilde{Y})) - \tilde{o}(C_*(\tilde{Z})) = 0 - 0 = 0. \end{aligned}$$

Lemma 2.36 (iii) implies that Q_* is $\mathbb{Z}\pi$ -chain homotopy equivalent to a finite free positive $\mathbb{Z}\pi$ -chain complex F_* . Choose a $\mathbb{Z}\pi$ -chain homotopy equivalence $g_*: \Sigma^3 F_* \rightarrow \text{cone}_*(C_*(\tilde{h}))$. We get a commutative diagram of $\mathbb{Z}\pi$ -chain complexes with exact rows and $\mathbb{Z}\pi$ -chain homotopy equivalences as vertical arrows

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_*(\tilde{Z}) & \longrightarrow & C_* & \longrightarrow & \Sigma^3 F_* \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow g_*' & & \downarrow g_* \\ 0 & \longrightarrow & C_*(\tilde{Z}) & \longrightarrow & \text{cyl}_*(C_*(\tilde{h})) & \longrightarrow & \text{cone}_*(C_*(\tilde{h})) \longrightarrow 0 \end{array}$$

by requiring that the right square is a pull back. Now define the desired $\mathbb{Z}\pi$ -chain map $f_*: C_* \rightarrow C_*(\tilde{Y})$ to be the composite of g'_* with the canonical $\mathbb{Z}\pi$ -chain homotopy equivalence $\text{cyl}_*(C_*(\tilde{h})) \rightarrow C_*(\tilde{Y})$. $\square$

Next we present the main tool to pass from chain complexes to CW -complexes. Its proof can be found in [984, Theorem 2] or in the more general equivariant setting in [644, Theorem 13.19 on page 268].

Theorem 2.46 (Realization Theorem). *Let $h: Z \rightarrow Y$ be a map between connected CW -complexes such that $\pi_1(h): \pi_1(Z) \rightarrow \pi_1(Y)$ is an isomorphism. In the sequel we identify $\pi = \pi_1(Y)$ with $\pi_1(Z)$ using $\pi_1(h)$. Put $d = \dim(Z)$ and suppose $2 \leq d < \infty$. Assume the existence of a free $\mathbb{Z}\pi$ -chain complex C_* with a preferred $\mathbb{Z}\pi$ -basis and a $\mathbb{Z}\pi$ -chain homotopy equivalence $f_*: C_* \rightarrow C_*(\tilde{Y})$ such that the restriction $C_*|_d$ to dimensions $0, 1, \dots, d$ agrees with $C_*(\tilde{Z})$ and $f_*|_d = C_*(\tilde{h})$.*

Then we can construct a CW -complex X such that its d -skeleton X_d agrees with Z and a cellular homotopy equivalence $g: X \rightarrow Z$ satisfying under the obvious identification $\pi = \pi_1(X) = \pi_1(Y) = \pi_1(Z)$:

- (i) *We have $g|_Z = h$;*
- (ii) *There is a $\mathbb{Z}\pi$ -chain isomorphism $u_*: C \xrightarrow{\cong} C_*(\tilde{X})$ such that the given $\mathbb{Z}\pi$ -basis on C_* is mapped bijectively to the cellular $\mathbb{Z}\pi$ -basis of $\tilde{X}$;*
- (iii) *We have $C_*(g) \circ u_* = f_*$.*

Remark 2.47. Note that there is no absolute version of the Realization Theorem 2.46 in the sense that, for a d -dimensional CW -complex Z with fundamental group π and dimension $d \geq 2$ and a based free $\mathbb{Z}\pi$ -chain complex C_* with $C_*|_d = C_*(\tilde{Z})$, we can find a CW -complex X with $X_d = Z$ and $C_*(\tilde{X}) = C_*$. Moreover, the assumption $\dim(Z) \geq 2$ cannot be dropped in the Realization Theorem 2.46.

Lemma 2.48. *Let X be a connected CW -complex. Then it is finitely dominated if and only if $\pi_1(X)$ is finitely presented and the $\mathbb{Z}[\pi_1(X)]$ -chain complex $C_*(\tilde{X})$ is finitely dominated.*

Proof. This follows essentially from Theorem 2.46; details of the proof can be found in [984, Corollary 5.1] or in the more general equivariant setting in [644, Proposition 14.6 (a) on page 282]. $\square$

Next we can give the proof of Theorem 2.39.

Proof of Theorem 2.39. (i) If the finitely dominated connected CW -complex Y is homotopy equivalent to a finite CW -complex, we get $\tilde{o}(Y) = 0$ directly from the definitions. Now suppose that Y is a finitely dominated connected CW -complex with $\tilde{o}(Y) = 0$. We conclude from Lemma 2.45 and Theorem 2.46 that Y is homotopy equivalent to a CW -complex X for which its cellular $\mathbb{Z}\pi$ -chain complex $C_*(\tilde{X})$ is finite free. The latter implies that X is finite.

(ii). Since G is finitely presented, we can choose a connected finite 2-dimensional CW -complex Z with $\pi_1(Z) = G$. Consider any element $\xi \in \tilde{K}_0(\mathbb{Z}\pi)$. Choose a finitely generated projective R -module P and a natural number n such that $\xi = [P] - [\mathbb{Z}\pi^n]$ holds. Choose an exact sequence $0 \rightarrow \bigoplus_{I_3} \mathbb{Z}\pi \xrightarrow{u} \bigoplus_{I_2} \mathbb{Z}\pi \rightarrow P \rightarrow 0$. Now consider $X' = X \vee \bigvee_{i_2 \in I} S^2$. For each $i_3 \in I_3$ we attach a 3-cell to X' with an attaching map $q_{i_3}: S^2 \rightarrow X'$ such that $[q_{i_3}] \in \pi_2(X')$ corresponds to the image of the basis element in $\bigoplus_{I_3} \mathbb{Z}\pi$ associated to i_3 under the composite

$$\bigoplus_{I_3} \mathbb{Z}\pi \xrightarrow{u} \bigoplus_{I_2} \mathbb{Z}\pi \xrightarrow{j} \pi_2(X')$$

where j sends the basis element associated to $i_2 \in I_2$ to the element in $\pi_2(X')$ given by the obvious inclusion of $S^2 \rightarrow X'$ associated to i_2 . Call the resulting 3-dimensional CW -complex Y . Note that we can identify π with $\pi_1(Y)$. We obtain an exact sequence of free $\mathbb{Z}\pi$ -chain complexes

$$0 \rightarrow C_*(\tilde{X}) \rightarrow C_*(\tilde{Y}) \rightarrow C_*(\tilde{Y}, \tilde{X}) \rightarrow 0.$$

The $\mathbb{Z}\pi$ -chain complex $C_*(\tilde{Y}, \tilde{X})$ is concentrated in dimensions 2 and 3 and its third differential is u . This implies that $C_*(\tilde{Y}, \tilde{X})$ is $\mathbb{Z}\pi$ -chain homotopy equivalent to the $\mathbb{Z}\pi$ -chain complex concentrated in dimension 2 with P as second $\mathbb{Z}\pi$ -chain module. Hence $C_*(\tilde{Y}, \tilde{X})$ is finitely dominated and $o(C_*(\tilde{Y}, \tilde{X})) = [P]$ by Lemma 2.36 (i). Lemma 2.36 (ii) implies that $C_*(\tilde{Y})$ is finitely dominated. Then Y is finitely dominated as a CW -complex by Lemma 2.48. Lemma 2.36 (ii) implies that we get for some integer m

$$o(C_*(\tilde{Y})) = o(C_*(\tilde{Z})) + o(C_*(\tilde{Y}, \tilde{X})) = m \cdot [\mathbb{Z}\pi] + [P].$$

By attaching to Y finitely many trivial 2 and 3-cells, we can arrange that Y is a finitely dominated connected CW -complex with $\pi_1(Y) = G$ and $o(Y) = [P] - [\mathbb{Z}\pi^n] = \xi$. $\square$

Exercise 2.49. Let

$$\begin{array}{ccc} X_0 & \xrightarrow{i_1} & X_1 \\ i_2 \downarrow & \searrow j_0 & \downarrow j_1 \\ X_2 & \xrightarrow{j_2} & X \end{array}$$

be a cellular pushout, i.e., the diagram is a pushout, the map i_1 is an inclusion of CW -complexes, the map i_2 is cellular and X carries the induced CW -structure. Suppose that X_0, X_1, X_2 are finitely dominated.

Then X is finitely dominated and we get in $K_0(\mathbb{Z}[\pi_1(X)])$

$$o(X) = (j_1)_*(o(X_1)) + (j_2)_*(o(X_2)) - (j_0)_*(o(X_1)).$$

2.6 Geometric Interpretation of Projective Class Group and Finiteness Obstruction

Next we give a geometric construction of $\widetilde{K}_0(\mathbb{Z}\pi)$ that is in the spirit of the well-known interpretation of the Whitehead group in terms of deformation retractions, which we will present later in Section 3.4. The material of this section is taken from [642], where more information and details of the proofs can be found.

Given a space Y , we want to define an abelian group $\text{Wa}(Y)$. The underlying set is the set of equivalence classes of an equivalence relation $\sim$ defined on the set of maps $f: X \rightarrow Y$ with finitely dominated CW -complexes as source and the given space Y as target. We call $f_0: X_0 \rightarrow Y$ and $f_4: X_4 \rightarrow Y$ equivalent if there exists a commutative diagram

$$\begin{array}{ccccccc}
 X_0 & \xrightarrow{i_0} & X_1 & \xrightarrow{j_1} & X_2 & \xleftarrow{j_3} & X_3 & \xleftarrow{i_4} & X_4 \\
 & \searrow f_0 & & \searrow f_1 & \downarrow f_2 & \swarrow f_3 & & \swarrow f_4 & \\
 & & & & Y & & & &
 \end{array}$$

such that j_1 and j_3 are homotopy equivalences and i_0 and i_4 are inclusions of CW -complexes with the property that the larger one is obtained from the smaller one by attaching finitely many cells. Obviously this relation is symmetric and reflexive. It needs some work to show transitivity and hence that it is an equivalence relation. The addition in $\text{Wa}(Y)$ is given by the disjoint sum, i.e., define the sum of the class of $f_0: X_0 \rightarrow Y$ and $f_1: X_1 \rightarrow Y$ to be the class of $f_0 \amalg f_1: X_0 \amalg X_1 \rightarrow Y$. It is easy to check that this is compatible with the equivalence relation. The neutral element is represented by $\emptyset \rightarrow Y$. The inverse of the class $[f]$ of $f: X \rightarrow Y$ is constructed as follows. Choose a finite domination (Z, i, r) of X . Construct a map $F: \text{cyl}(i) \rightarrow X$ from the mapping cylinder of i to Y such that $F|_X = \text{id}_X$ and $F|_Z = r$. Then an inverse of $[f]$ is given by the class $[f']$ of the composite

$$f': \text{cyl}(i) \cup_X \text{cyl}(i) \xrightarrow{F \cup_{\text{id}_X} F} X \xrightarrow{f} Y.$$

This finishes the definition of the abelian group $\text{Wa}(Y)$. A map $f: Y_0 \rightarrow Y_1$ induces a homomorphism of abelian groups $\text{Wa}(f): \text{Wa}(Y_0) \rightarrow \text{Wa}(Y_1)$ by composition. Thus Wa defines a functor from the category of spaces to the category of abelian groups.

Exercise 2.50. Show that $[f] + [f'] = 0$ holds for the composite f' above.

Given a finitely dominated CW -complex X , define its *geometric finiteness obstruction* $o_{\text{geo}}(X) \in \text{Wa}(X)$ by the class of id_X .

Theorem 2.51 (The geometric finiteness obstruction). *Let X be a finitely dominated CW -complex. Then X is homotopy equivalent to a finite CW -complex if and only if $o_{\text{geo}}(X) = 0$ in $\text{Wa}(X)$.*

Proof. Obviously $o_{\text{geo}}(X) = 0$ if X is homotopy equivalent to a finite CW -complex. Suppose $o_{\text{geo}}(X) = 0$. Hence there are a CW -complex Y , a map $r: Y \rightarrow X$ and a homotopy equivalence $h: Y \rightarrow Z$ to a finite CW -complex Z such that Y is obtained from X by attaching finitely many cells and $r \circ i = \text{id}_X$ holds for the inclusion $i: X \rightarrow Y$. The mapping cylinder $\text{cyl}(r)$ is built from the mapping cylinder $\text{cyl}(i)$ by attaching a finite number of cells and is homotopy equivalent to X . Choose a homotopy equivalence $g: \text{cyl}(i) \rightarrow Z$. Consider the push-out

$$\begin{array}{ccc} \text{cyl}(i) & \xrightarrow{i} & \text{cyl}(r) \\ g \downarrow & & \downarrow g' \\ Z & \xrightarrow{i'} & Z' \end{array}$$

where i is the inclusion. Since g is a homotopy equivalence, the same is true for g' . Hence X is homotopy equivalent to the finite CW -complex Z' . $\square$

Theorem 2.52 (Identifying the finiteness obstruction with its geometric counterpart). *Let Y be a space. Then there is a natural isomorphism of abelian groups*

$$\Phi: \text{Wa}(Y) \xrightarrow{\cong} \bigoplus_{C \in \pi_0(Y)} \widetilde{K}_0(\mathbb{Z}\pi_1(C)).$$

Proof. We only explain the definition of Φ . Consider an element $[f] \in \text{Wa}(Y)$ represented by a map $f: X \rightarrow Y$ from a finitely dominated CW -complex X to Y . Given a path component C of X , let C_f be the path component of Y containing $f(C)$. The map f induces a map $f|_C: C \rightarrow C_f$ and hence a map $(f|_C)_*: \widetilde{K}_0(\mathbb{Z}\pi_1(C)) \rightarrow \widetilde{K}_0(\mathbb{Z}\pi_1(C_f))$. Since X is finitely dominated, every path component C of X is finitely dominated, and we can consider its finiteness obstruction $\tilde{o}(C) \in \widetilde{K}_0(\mathbb{Z}\pi_1(C))$. Let $\phi([f])_C$ be the image of $\tilde{o}(C)$ under the composite

$$\widetilde{K}_0(\mathbb{Z}\pi_1(C)) \xrightarrow{(f|_C)_*} \widetilde{K}_0(\mathbb{Z}\pi_1(C_f)) \rightarrow \bigoplus_{C \in \pi_0(Y)} \widetilde{K}_0(\mathbb{Z}\pi_1(C)).$$

Since $\pi_0(X)$ is finite, we can define

$$\phi([f]) := \sum_{C \in \pi_0(X)} \phi([f])_C.$$

We omit the easy proof that this is compatible with the equivalence relation appearing in the definition of $\text{Wa}(Y)$, that ϕ is a homomorphism of abelian groups and that Theorem 2.39 implies that Φ is bijective. $\square$

2.7 Universal Functorial Additive Invariants

In this section we describe the pair $(K_0(\mathbb{Z}\pi_1(X)), o(X))$ by an abstract property.

Definition 2.53 (Functorial additive invariant for finitely dominated CW-complexes). A *functorial additive invariant for finitely dominated CW-complexes* consists of a covariant functor A from the category of finitely dominated CW-complexes to the category of abelian groups together with an assignment a that associates to every finitely dominated CW-complex X an element $a(X) \in A(X)$ such that the following axioms are satisfied:

- Homotopy invariance of A
If $f, g: X \rightarrow Y$ are homotopic maps between finitely dominated CW-complexes, then $A(f) = A(g)$;
- Homotopy invariance of $a(X)$
If $f: X \rightarrow Y$ is a homotopy equivalence of finitely dominated CW-complexes, then $A(f)(a(X)) = a(Y)$;
- Additivity
Let

$$\begin{array}{ccc} X_0 & \xrightarrow{i_1} & X_1 \\ i_2 \downarrow & \searrow j_0 & \downarrow j_1 \\ X_2 & \xrightarrow{j_2} & X \end{array}$$

be a *cellular pushout*, i.e., the diagram is a pushout, the map i_1 is an inclusion of CW-complexes, the map i_2 is cellular and X carries the induced CW-structure. Suppose that X_0, X_1, X_2 are finitely dominated. Then X is finitely dominated and

$$a(X) = A(j_1)(a(X_1)) + A(j_2)(a(X_2)) - A(j_0)(a(X_0));$$

- Normalization
 $a(\emptyset) = 0$.

Example 2.54 (Componentwise Euler characteristic). Let A be the covariant functor sending a finitely dominated CW-complex X to $H_0(X; \mathbb{Z}) = \bigoplus_{C \in \pi_0(X)} \mathbb{Z}$. Let $a(X) \in A(X)$ be the componentwise Euler characteristic, i.e., the collection of integers $\{\chi(C) \mid C \in \pi_0(X)\}$. Then (A, a) is a functorial additive invariant for finitely dominated CW-complexes.

Definition 2.55 (Universal functorial additive invariant for finitely dominated CW-complexes). A *universal functorial additive invariant for finitely dominated CW-complexes* (U, u) is a functorial additive invariant with the property that for any functorial additive invariant (A, a) there is precisely one natural transformation $T: U \rightarrow A$ with the property that $T(X)(u(X)) = a(X)$ holds for every finitely dominated CW-complex X .

Exercise 2.56. Show that the functorial additive invariant defined in Example 2.54 is the universal one if we restrict to finite CW -complexes.

Obviously the universal additive functorial invariant is unique (up to unique natural equivalence) if it exists. It is also easy to construct it. However, it turns out that there exists a concrete model, namely, the following theorem is proved in [642, Theorem 4.1].

Theorem 2.57 (The finiteness obstruction is the universal functorial additive invariant). *The covariant functor $X \mapsto \bigoplus_{C \in \pi_0(X)} K_0(\mathbb{Z}\pi_1(C))$ together with the componentwise finiteness obstruction $\{o(C) \mid C \in \pi_0(X)\}$ is the universal functorial additive invariant for finitely dominated CW -complexes.*

Exercise 2.58. (i) Construct for finitely dominated CW -complexes X and Y a natural bilinear pairing

$$P(X, Y): U(X) \times U(Y) \rightarrow U(X \times Y)$$

sending $(u(X), u(Y))$ to $u(X \times Y)$ where (U, u) is the universal functorial additive invariant for finitely dominated CW -complexes;

(ii) Let X be a finitely dominated CW -complex. Let Y be a finite CW -complex such that $\chi(C) = 0$ for every component C of Y . Show that $X \times Y$ is homotopy equivalent to a finite CW -complex.

2.8 Variants of the Farrell-Jones Conjecture for $K_0(RG)$

In this section we state variants of the Farrell-Jones Conjecture for $K_0(RG)$, where RG , sometimes also written as $R[G]$, is the *group ring* of a group G with coefficients in an associative ring R with unit. Elements in RG are given by formal finite sums $\sum_{g \in G} r_g \cdot g$, and addition and multiplication is given by

$$\begin{aligned} \left(\sum_{g \in G} r_g \cdot g \right) + \left(\sum_{g \in G} s_g \cdot g \right) &:= \sum_{g \in G} (r_g + s_g) \cdot g; \\ \left(\sum_{g \in G} r_g \cdot g \right) \cdot \left(\sum_{g \in G} s_g \cdot g \right) &:= \sum_{g \in G} \left(\sum_{\substack{h, k \in G, \\ g = hk}} r_h \cdot s_k \right) \cdot g. \end{aligned}$$

The Farrell-Jones Conjecture itself will give a complete answer for arbitrary groups and rings, but to formulate the full version some additional effort will be needed. If one assumes that R is regular and G is torsionfree or that R is regular and $\mathbb{Q} \subseteq R$, then the conjecture reduces to easy to formulate statements, which we will present next. Moreover, these special cases are already very interesting.

Definition 2.59 (Projective resolution). Let M be an R -module. A *projective resolution* (P_*, ϕ) of M is a positive projective R -chain complex P_* with $H_n(P_*) = 0$ for $n \geq 1$ together with an R -isomorphism $\phi: H_0(P_*) \xrightarrow{\cong} M$. It is called *finite*, *finitely*

generated, free, finite-dimensional, or d -dimensional if the R -chain complex P_* has this property.

A ring R is *Noetherian* if any submodule of a finitely generated R -module is again finitely generated. A ring R is called *regular* if it is Noetherian and any finitely generated R -module has a finite-dimensional projective resolution. Any principal ideal domain such as $\mathbb{Z}$, any field, and, more generally, any Dedekind domain is regular, see Theorem 2.21.

Conjecture 2.60 (Farrell-Jones Conjecture for $K_0(R)$ for torsionfree G and regular R). Let G be a torsionfree group and let R be a regular ring. Then the map induced by the inclusion of the trivial group into G

$$K_0(R) \xrightarrow{\cong} K_0(RG)$$

is bijective.

In particular we get for any principal ideal domain R and torsionfree G

$$\tilde{K}_0(RG) = 0.$$

Remark 2.61 (Relevance of Conjecture 2.60). In view of Remark 2.8 Conjecture 2.60 is equivalent to the statement that for a torsionfree group G and a regular ring R every finitely generated projective RG -module is stably finitely generated free. This is the algebraic relevance of this conjecture. Its geometric meaning comes from the following conclusion of Theorem 2.39. Namely, if $R = \mathbb{Z}$ and G is a finitely presented torsionfree group, it is equivalent to the statement that every finitely dominated CW -complex with $\pi_1(X) \cong G$ is homotopy equivalent to a finite CW -complex.

Definition 2.62 (Family of subgroups). A family $\mathcal{F}$ of subgroups of a group G is a set of subgroups that is closed under conjugation with elements of G and under passing to subgroups.

Our main examples of families are listed below

Notation 2.63.

notation	subgroups
$\mathcal{TR}$	trivial group
$\mathcal{FCY}$	finite cyclic subgroups
$\mathcal{FIN}$	finite subgroups
$\mathcal{CYC}$	cyclic subgroups
$\mathcal{VCY}$	virtually cyclic subgroups
$\mathcal{ALL}$	all subgroups

Definition 2.64 (Orbit category). The orbit category $\text{Or}(G)$ has as objects homogeneous spaces G/H and as morphisms G -maps. Given a family $\mathcal{F}$ of subgroups of G , let the $\mathcal{F}$ -restricted orbit category $\text{Or}_{\mathcal{F}}(G)$ be the full subcategory of $\text{Or}(G)$ whose objects are homogeneous spaces G/H with $H \in \mathcal{F}$.

Definition 2.65 (Subgroup category). The *subgroup category* $\text{Sub}(G)$ has as objects subgroups H of G . For $H, K \subseteq G$, let $\text{conhom}_G(H, K)$ be the set of all group homomorphisms $f: H \rightarrow K$ for which there exists a group element $g \in G$ such that f is given by conjugation with g . The group of inner automorphisms $\text{inn}(K)$ consists of those automorphisms $K \rightarrow K$ that are given by conjugation with an element $k \in K$. It acts on $\text{conhom}(H, K)$ from the left by composition. Define the set of morphisms in $\text{Sub}(G)$ from H to K to be $\text{inn}(K) \backslash \text{conhom}(H, K)$. Composition of group homomorphisms defines the composition of morphisms in $\text{Sub}(G)$.

Given a family $\mathcal{F}$, define the $\mathcal{F}$ -restricted category of subgroups $\text{Sub}_{\mathcal{F}}(G)$ to be the full subcategory of $\text{Sub}(G)$ that is given by objects H belonging to $\mathcal{F}$.

Exercise 2.66. Show that $\text{Sub}_{\mathcal{F}}(G)$ is a quotient category of $\text{Or}_{\mathcal{F}}(G)$.

Note that there is a morphism from H to K only if H is conjugate to a subgroup of K . Clearly $K_0(R(-))$ yields a functor from $\text{Sub}_{\mathcal{F}}(G)$ to abelian groups since inner automorphisms on a group K induce the identity on $K_0(RK)$. Using the inclusions into G , one obtains a map

$$\text{colim}_{H \in \text{Sub}_{\mathcal{F}}(G)} K_0(RH) \rightarrow K_0(RG).$$

We briefly recall the notion of a colimit of a covariant functor $F: C \rightarrow \mathbb{Z}\text{-MOD}$ from a small category C into the category of abelian groups, where *small* means that the objects of C form a set. Given an abelian group A , let C_A be the constant functor $C \rightarrow \mathbb{Z}\text{-MOD}$ that sends every object in C to A and every morphism in C to id_A . Given a homomorphism $f: A \rightarrow B$ of abelian groups, let $C_f: C_A \rightarrow C_B$ be the obvious transformation. The *colimit*, sometimes also called the *direct limit*, of F consists of an abelian group $\text{colim}_C F$ together with a transformation $T_F: F \rightarrow C_{\text{colim}_C F}$ such that for any abelian group B and transformation $T: F \rightarrow C_B$ there exists precisely one homomorphism of abelian groups $\phi: \text{colim}_C F \rightarrow B$ satisfying $C_\phi \circ T_F = T$. The colimit is unique (up to unique isomorphism) and always exists. If we replace abelian group by ring or by R -module respectively, we get the notion of a *colimit*, sometimes also called a *direct limit*, of functors from a small category to rings or R -modules respectively.

Conjecture 2.67 (Farrell-Jones Conjecture for $K_0(RG)$ for regular R with $\mathbb{Q} \subseteq R$). Let R be a regular ring with $\mathbb{Q} \subseteq R$ and let G be a group.

Then the homomorphism

$$(2.68) \quad I_{\mathcal{FIN}}(G, F): \text{colim}_{H \in \text{Sub}_{\mathcal{FIN}}(G)} K_0(RH) \rightarrow K_0(RG)$$

coming from the various inclusions of finite subgroups of G into G is a bijection.

One can also ask for the following stronger version of Conjecture 2.67, which also encompasses Conjecture 2.60.

Conjecture 2.69 (Farrell-Jones Conjecture for $K_0(RG)$ for regular R). Let R be a regular ring and let G be a group. Let $\mathcal{P}(G, R)$ be the set of primes which are not invertible in R and for which G contains an element of order p .

Then the homomorphism

$$I_{\mathcal{FIN}}(G, F): \operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{FIN}}(G)} K_0(RH) \rightarrow K_0(RG)$$

coming from the various inclusions of finite subgroups of G into G is a $\mathcal{P}(G, R)$ -isomorphism, i.e., an isomorphism after inverting all primes in $\mathcal{P}(G, R)$.

We mention that the surjectivity of the map $I_{\mathcal{FIN}}(G, F)$ is equivalent to the surjectivity of the map induced by the various inclusions of subgroups $H \in \mathcal{FIN}$ into G

$$\bigoplus_{H \in \mathcal{FIN}} K_0(RH) \rightarrow K_0(RG),$$

because this map factorizes as

$$\bigoplus_{H \in \mathcal{FIN}} K_0(RH) \xrightarrow{\psi} \operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{FIN}}(G)} K_0(RH) \xrightarrow{I_{\mathcal{FIN}}(G, F)} K_0(RG),$$

where the first map ψ is surjective.

Remark 2.70 (Module-theoretic relevance of Conjecture 2.67). Conjecture 2.67 implies that for a regular ring R with $\mathbb{Q} \subseteq R$ every finitely generated projective R -module is, up to adding finitely generated free RG -modules, a direct sum of finitely many RG -modules of the shape $RG \otimes_{RH} P$ for a finite subgroup $H \subseteq G$ and a finitely generated projective RH -module P . So it predicts the (stable) structure of finitely generated projective RG -modules in the most elementary way. We mention, however, that the situation is much more complicated in the case where we drop the assumption that R is regular and $\mathbb{Q} \subseteq R$. In particular, for $R = \mathbb{Z}$ new phenomena will occur, as explained later, which are related to so-called negative K -groups and Nil-groups. For instance, the obvious inclusion $\mathbb{Z}/6 \rightarrow \mathbb{Z} \times \mathbb{Z}/6$ does not induce a surjection $K_0(\mathbb{Z}[\mathbb{Z}/6]) \rightarrow K_0(\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}/6])$, since $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/6]) = 0$ and $\tilde{K}_0(\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}/6]) \cong \mathbb{Z}$, whereas by $K_0(\mathbb{Q}[\mathbb{Z}/6]) \rightarrow K_0(\mathbb{Q}[\mathbb{Z} \times \mathbb{Z}/6])$ is known to be bijective as predicted by Conjecture 2.67.

Remark 2.71 (Conjecture 2.67 and the Atiyah Conjecture). Conjecture 2.67 plays a role in a program aiming at a proof of the Atiyah Conjecture about L^2 -Betti numbers, as explained in [650, Section 10.2]. Atiyah defined the n -th L^2 -Betti number of the universal covering $\tilde{M}$ of a closed Riemannian manifold M to be the non-negative real number

$$b_n^{(2)}(\tilde{M}) := \lim_{t \rightarrow \infty} \int_{\mathcal{F}} \operatorname{tr}(e^{-t\Delta_n(\tilde{x}, \tilde{x})}) d\tilde{x}$$

where $\mathcal{F}$ is a fundamental domain for the $\pi_1(M)$ -action and $e^{-t\Delta_n(\tilde{x}, \tilde{x})}$ denotes the heat kernel on $\tilde{M}$. The version of the Atiyah Conjecture which we are interested in and which is at the time of writing open says that $d \cdot b_n^{(2)}(\tilde{M})$ is an integer if d is an integer such that the order of any finite subgroup of $\pi_1(M)$ divides d . In particular $b_n^{(2)}(\tilde{M})$ is expected to be an integer if $\pi_1(M)$ is torsionfree. This gives an interesting

connection between the analysis of heat kernels and the projective class group of complex group rings $\mathbb{C}G$.

If one drops the condition that there exists a bound on the order of finite subgroups of $\pi_1(M)$, then also transcendental real numbers can occur as the L^2 -Betti number of the universal covering $\tilde{M}$ of a closed Riemannian manifold M , see [58, 433, 809].

An R -module M is called *Artinian* if for any descending series of submodules $M_1 \supseteq M_2 \supseteq \dots$ there exists an integer k such that $M_k = M_{k+1} = M_{k+2} = \dots$ holds. An R -module M is called *simple* or *irreducible* if $M \neq \{0\}$ and M contains only $\{0\}$ and M as submodules. A ring R is called *Artinian* if both R considered as a left R -module is Artinian and R considered as a right R -module is Artinian, or, equivalently, every finitely generated left R -module and every finitely generated right R -module is Artinian. Skew-fields and finite rings are Artinian, whereas $\mathbb{Z}$ is not Artinian.

Conjecture 2.72 (Farrell-Jones Conjecture for $K_0(RG)$ for an Artinian ring R).

Let G be a group and R be an Artinian ring.

Then the canonical map

$$I_{\mathcal{FIN}}(G, R): \operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{FIN}}(G)} K_0(RH) \rightarrow K_0(RG)$$

is an isomorphism

2.9 Kaplansky's Idempotent Conjecture

In this section we discuss the following conjecture.

Conjecture 2.73 (Kaplansky's Idempotent Conjecture). Let R be an integral domain and let G be a torsionfree group. Then all idempotents of RG are trivial, i.e., equal to 0 or 1.

Remark 2.74 (Kaplansky's Idempotent Conjecture for prime characteristic).

There is a reasonable more general version of Conjecture 2.73 where one replaces the condition that G is torsionfree by the weaker condition that any prime p which divides the order of some finite subgroup $H \subseteq G$ is not invertible in the integral domain R . If R is a skew-field of prime characteristic p , then this condition reduces to the condition that any finite subgroup H of G is a p -group.

The version of Kaplansky's Idempotent Conjecture 2.73 described in Remark 2.74 is consistent with the observation that the only known idempotents in a group ring RG come from idempotents in R or by the following construction.

Example 2.75 (Construction of idempotents). Let G be a group and $g \in G$ be an element of finite order. Suppose that the order $|g|$ is invertible in R . Define an element $x := |g|^{-1} \cdot \sum_{i=1}^{|g|} g^i$. Then $x^2 = x$, i.e., x is an idempotent in RG .

Exercise 2.76. Show that the version of Kaplansky's Idempotent Conjecture of Remark 2.74 holds for $G = \mathbb{Z}/2$.

Exercise 2.77. Consider the ring $R = \mathbb{Z}[x]/(2x^2 - 3x + 1)$. In the sequel denote by $\bar{u}$ the class of $u \in \mathbb{Z}[x]$ in R . Show:

- (i) 2 is not invertible in R ;
- (ii) There are precisely two non-trivial idempotents in R , namely $\overline{2 - 2x}$ and $\overline{-1 + 2x}$;
- (iii) The element $\bar{x} + (1 - \bar{x}) \cdot t$ is a non-trivial idempotent in $R[\mathbb{Z}/2]$.

Remark 2.78 (Sofic groups). In the next theorem we will use the notion of a *sofic group* that was introduced by Gromov and originally called *subamenable group*. Every residually amenable group is sofic but the converse is not true. The class of sofic groups is closed under taking subgroups, direct products, amalgamated free products, colimits and inverse limits, and, if H is a sofic normal subgroup of G with amenable quotient G/H , then G is sofic. To the authors' knowledge there is no example of a group that is not sofic. There is a note by Dave Witte Morris [752] following Deligne [300] where a central extension $1 \rightarrow \mathbb{Z} \rightarrow G \rightarrow SP(2n, \mathbb{R}) \rightarrow 1$ is constructed such that G is not residually finite. The group G is viewed as a candidate for a group which is not sofic. It is unknown but likely to be true that all hyperbolic groups are sofic. For more information about the notion of a sofic group we refer to [332].

Definition 2.79 (Directly finite). An R -module M is called *directly finite* if every R -module N satisfying $M \cong_R M \oplus N$ is trivial. A ring R is called *directly finite* (or *von Neumann finite*) if it is directly finite as a module over itself, or, equivalently, if $r, s \in R$ satisfy $rs = 1$, then $sr = 1$. A ring is called *stably finite* if the matrix algebra $M_n(R)$ is directly finite for all $n \geq 1$.

Remark 2.80 (Stable finiteness). Stable finiteness for a ring R is equivalent to the following statement. Every finitely generated projective R -module P whose class in $K_0(R)$ is zero is already the trivial module, i.e., $0 = [P] \in K_0(R)$ implies $P \cong 0$.

If F is a field of characteristic zero, then FG is stably finite for every group G . This is proved by Kaplansky [544], see also Passman [791, Corollary 1.9 on page 38]. If R is a skew-field and G is a sofic group, then RG is stably finite. This is proved for free-by-amenable groups by Ara-Meara-Perera [35] and extended to sofic groups by Elek-Szabo [331, Corollary 4.7]. These results have been extended to extensions with a finitely generated residually finite groups as kernel and a sofic finitely generated group as quotient by Berlai [128].

The next theorem is taken from [88, Theorem 1.12].

Theorem 2.81 (The Farrell-Jones Conjecture and Kaplansky's Idempotent Conjecture). Let G be a group. Let R be a ring whose idempotents are all trivial. Suppose that

$$K_0(R) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_0(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism.

Then 0 and 1 are the only idempotents in RG if one of the following conditions is satisfied:

- (i) RG is stably finite;
- (ii) R is a field of characteristic zero;
- (iii) R is a skew-field and G is sofic.

Remark 2.82 (The Farrell-Jones Conjecture and Kaplansky's Idempotent Conjecture). Theorem 2.81 implies that for a skew-field D of characteristic zero and a torsionfree group G Kaplansky's Idempotent Conjecture 2.73 is true for DG , provided that Conjecture 2.60 holds and that D is commutative or G is sofic.

Remark 2.83 (The Farrell-Jones Conjecture and the Kaplansky's Idempotent Conjecture for prime characteristic). Suppose that D is a skew-field of prime characteristic p , that Conjecture 2.72 holds for G and D , and that all finite subgroups of G are p -groups. Then $K_0(D) \xrightarrow{\cong} K_0(DG)$ is an isomorphism since for a finite p -group H the group ring DH is a local ring, see [271, Theorem 5.24 on page 114], and hence $\tilde{K}_0(DH) = 0$ by Lemma 2.123. If we furthermore assume that G is sofic, then Theorem 2.81 implies that all idempotents in DG are trivial.

Remark 2.84 (Reducing the case of a field of characteristic zero to $\mathbb{C}$). Let F be a field of characteristic zero and let $u = \sum_{g \in G} x_g \cdot g \in FG$ be an element. Let K be the finitely generated field extension of $\mathbb{Q}$ given by $K = \mathbb{Q}(x_g \mid g \in G) \subset F$. Then u is already an element in KG . The field K embeds into $\mathbb{C}$ since K is finitely generated, it is a finite algebraic extension of a transcendental extension K' of $\mathbb{Q}$, see [617, Theorem 1.1 on p. 356], and K' has finite transcendence degree over $\mathbb{Q}$. Since the transcendence degree of $\mathbb{C}$ over $\mathbb{Q}$ is infinite, there exists an embedding $K' \hookrightarrow \mathbb{C}$ induced by an injection of a transcendence basis of K over $\mathbb{Q}$ into a transcendence basis of $\mathbb{C}$ over $\mathbb{Q}$. It extends to an embedding $K \hookrightarrow \mathbb{C}$ because $\mathbb{C}$ is algebraically closed. Hence u can be viewed as an element in $\mathbb{C}G$. This reduces the case of fields F of characteristic zero to the case $F = \mathbb{C}$.

Next we mention some further results.

Formanek [398, Theorem 9], see also [189, Proposition 4.2], has shown that all idempotents of FG are trivial, provided that F is a field of characteristic zero and there are infinitely many primes p for which there do not exist an element $g \in G, g \neq 1$ and an integer $k \geq 1$ such that g and g^{p^k} are conjugate. Torsionfree hyperbolic groups satisfy these conditions. Hence Formanek's results imply that all idempotents in FG are trivial if G is torsionfree hyperbolic and F is a field of characteristic zero.

Delzant [301] has proved the Kaplansky's Idempotent Conjecture 2.73 for all integral domains R for a torsionfree hyperbolic group G , provided that G admits an appropriate action with large enough injectivity radius. Delzant actually deals with zero-divisors and units as well.

Remark 2.85 (Conjectures related to the Idempotent conjecture). There are also the *Zero-Divisor Conjecture* due to Kaplansky, which predicts for an integral domain R and a torsionfree group G that RG has no non-trivial zero-divisors, and the *Embedding Conjecture* due to Malcev, which predicts for an integral domain R and a torsionfree group G that RG can be embedded into a skew-field. Obviously the Embedding Conjecture implies the Zero-Divisor Conjecture, which in turn implies the Idempotent Conjecture 2.73. The Zero-Divisor Conjecture does *not* follow from Conjecture 2.60. For a ring R with $\mathbb{Q} \subseteq R = \mathbb{C}$ the Zero-Divisor Conjecture follows from the *Atiyah Conjecture* about the integrality of L^2 -Betti numbers for torsionfree groups, see [650, Lemma 10.15 on page 376]. There is also the *Unit-Conjecture* 3.125, which implies the Zero-Divisor Conjecture, see [610, (6.20) on page 95], and is discussed in Section 3.14.

2.10 The Bass Conjectures

2.10.1 The Bass Conjecture for Fields of Characteristic Zero as Coefficients

Let G be a group. Let $\text{con}(G)$ be the set of conjugacy classes (g) of elements $g \in G$. Denote by $\text{con}(G)_f$ the subset of $\text{con}(G)$ consisting of those conjugacy classes (g) for which each representative g has finite order. Let R be a commutative ring. Let $\text{class}(G, R)$ and $\text{class}(G, R)_f$ be the free R -module with the set $\text{con}(G)$ and $\text{con}(G)_f$ as basis. This is the same as the R -module of R -valued functions on $\text{con}(G)$ and $\text{con}(G)_f$ with finite support. Define the *universal R -trace*

$$(2.86) \quad \text{tr}_{RG}^u : RG \rightarrow \text{class}(G, R), \quad \sum_{g \in G} r_g \cdot g \mapsto \sum_{g \in G} r_g \cdot (g).$$

It extends to a function $\text{tr}_{RG}^u : M_n(RG) \rightarrow \text{class}(G, R)$ on (n, n) -matrices over RG by taking the sum of the traces of the diagonal entries. Let P be a finitely generated projective RG -module. Choose a matrix $A \in M_n(RG)$ such that $A^2 = A$ and the image of the RG -map $r_A : RG^n \rightarrow RG^n$ given by right multiplication with A is RG -isomorphic to P . Define the *Hattori-Stallings rank* of P to be

$$(2.87) \quad \text{HS}_{RG}(P) = \text{tr}_{RG}^u(A) \in \text{class}(G, R).$$

The Hattori-Stallings rank depends only on the isomorphism class of the RG -module P . It induces an R -homomorphism, the *Hattori-Stallings homomorphism*,

$$(2.88) \quad \text{HS}_{RG} : K_0(RG) \otimes_{\mathbb{Z}} R \rightarrow \text{class}(G, R), \quad [P] \otimes r \mapsto r \cdot \text{HS}_{RG}(P).$$

Let F be a field of characteristic zero. Fix an integer $m \geq 1$. Let $F(\zeta_m) \supset F$ be the Galois extension given by adjoining the primitive m -th root of unity ζ_m to F . Denote by $\Gamma(m, F)$ the Galois group of this extension of fields, i.e., the group of automorphisms $\sigma : F(\zeta_m) \rightarrow F(\zeta_m)$ that induce the identity on F . It can be identified

with a subgroup of $\mathbb{Z}/m^*$ by sending σ to the unique element $u(\sigma) \in \mathbb{Z}/m^*$ for which $\sigma(\zeta_m) = \zeta_m^{u(\sigma)}$ holds. Let g_1 and g_2 be two elements of G of finite order. We call them *F-conjugate* if for some (and hence all) positive integers m with $g_1^m = g_2^m = 1$ there exists an element σ in the Galois group $\Gamma(m, F)$ with the property that $g_1^{u(\sigma)}$ and g_2 are conjugate. Two elements g_1 and g_2 are *F-conjugate* for $F = \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$, if the cyclic subgroups $\langle g_1 \rangle$ and $\langle g_2 \rangle$ are conjugate if g_1 and g_2 , or g_1 and g_2^{-1} , or g_1 and g_2 are conjugate, respectively.

Denote by $\text{con}_F(G)_f$ the set of *F-conjugacy classes* $(g)_F$ of elements $g \in G$ of finite order. Let $\text{class}_F(G)_f$ be the *F-vector space* with the set $\text{con}_F(G)_f$ as basis, or, equivalently, the *F-vector space* of functions $\text{con}_F(G)_f \rightarrow F$ with finite support. There are obvious inclusions of *F-modules*

$$\text{class}_F(G)_f \subseteq \text{class}(G, F)_f \subseteq \text{class}(G, F).$$

Lemma 2.89. *Suppose that F is a field of characteristic zero and H is a finite group. Then the Hattori-Stallings homomorphism, see (2.88), induces an isomorphism*

$$\text{HS}_{FH}: K_0(FH) \otimes_{\mathbb{Z}} F \xrightarrow{\cong} \text{class}_F(H)_f.$$

Proof. Since H is finite, an FH -module is a finitely generated projective FH -module if and only if it is a (finite-dimensional) H -representation with coefficients in F and $K_0(FH)$ is the same as the representation ring $\text{Rep}_F(H)$. The Hattori-Stallings rank $\text{HS}_{FH}(V)$ and the character χ_V of a G -representation V with coefficients in F are related by the formula

$$(2.90) \quad \chi_V(h^{-1}) = |C_G(h)| \cdot \text{HS}_{FH}(V)(h)$$

for $h \in H$ where $C_G(h)$ is the centralizer of h in G . Hence Lemma 2.89 follows from representation theory, see for instance [908, Corollary 1 in Chapter 12 on page 96]. $\square$

Exercise 2.91. Prove formula (2.90).

The following conjecture is the obvious generalization of Lemma 2.89 to infinite groups.

Conjecture 2.92 (Bass Conjecture for fields of characteristic zero as coefficients). Let F be a field of characteristic zero and let G be a group. The Hattori-Stallings homomorphism of (2.88) induces an isomorphism

$$\text{HS}_{FG}: K_0(FG) \otimes_{\mathbb{Z}} F \rightarrow \text{class}_F(G)_f.$$

Lemma 2.93. *Suppose that F is a field of characteristic zero and G is a group. Then the composite*

$$(2.94) \quad \operatorname{colim}_{H \in \operatorname{Sub}_{FIN}(G)} K_0(FH) \otimes_{\mathbb{Z}} F \xrightarrow{I_{FIN}(G,F) \otimes \operatorname{id}_F} K_0(FG) \otimes_{\mathbb{Z}} F \xrightarrow{\operatorname{HS}_{FG}} \operatorname{class}(G, F)$$

is injective and has as image $\operatorname{class}_F(G)_f$ where $I_{FIN}(G, F)$ is the map defined in (2.68).

Proof. This follows from the commutative diagram below, compare [646, Lemma 2.15 on page 220].

$$\begin{array}{ccc} \operatorname{colim}_{H \in \operatorname{Sub}_{FIN}(G)} K_0(FH) \otimes_{\mathbb{Z}} F & \xrightarrow{I_{FIN}(G,F) \otimes \operatorname{id}_F} & K_0(FG) \otimes_{\mathbb{Z}} F \\ \operatorname{colim}_{H \in \operatorname{Sub}_{FIN}(G)} \operatorname{HS}_{FH} \downarrow \cong & & \downarrow \operatorname{HS}_{FG} \\ \operatorname{colim}_{H \in \operatorname{Sub}_{FIN}(G)} \operatorname{class}_F(H)_f & \xrightarrow[\cong]{j} \operatorname{class}_F(G)_f \xrightarrow{i} & \operatorname{class}(G, F). \end{array}$$

Here the isomorphism j is the direct limit over the obvious maps $\operatorname{class}_F(H)_f \rightarrow \operatorname{class}_F(G)_f$ given by extending a class function in the trivial way and the map i is the natural inclusion and in particular injective. $\square$

Exercise 2.95. Let F be a field of characteristic zero. Show that the group G must be torsionfree if $\tilde{K}_0(FG)$ is a torsion group.

Theorem 2.96 (The Farrell-Jones Conjecture and the Bass Conjecture for fields of characteristic zero). *The Farrell-Jones Conjecture 2.67 for $K_0(RG)$ for regular R and $\mathbb{Q} \subseteq R$ implies the Bass Conjecture 2.92 for fields of characteristic zero as coefficients.*

Proof. This follows from Lemma 2.93. $\square$

The Bass Conjecture 14.23 implies the Bass Conjecture 2.92 for fields of characteristic zero as coefficients, provided that $F = \mathbb{C}$, see [131, Theorem 1.4 and Lemma 1.5].

Exercise 2.97. Let F be field of characteristic zero and let G be a group. Suppose that the Farrell-Jones Conjecture 2.67 for $K_0(RG)$ for regular R and $\mathbb{Q} \subseteq R$ holds for $R = F$. Consider any finitely generated projective FG -module P . Then the Hattori-Stallings rank $\operatorname{HS}_{FG}(P)$ evaluated at the unit $e \in G$ belongs to $\mathbb{Q} \subseteq F$.

Remark 2.98 (Zalesskii's Theorem). Zalesskii [1031], see also [189, Theorem 3.1], has shown for every field F , every group G , and every idempotent $x \in FG$ that $\operatorname{HS}_{FG}((x))$ evaluated at the unit $e \in G$ belongs to the prime field of F , where (x) is the finitely generated projective FG -module given by the two-sided ideal $(x) \subseteq FG$ spanned by x .

2.10.2 The Bass Conjecture for Integral Domains as Coefficients

Conjecture 2.99 (Bass Conjecture for integral domains as coefficients). Let R be a commutative integral domain and let G be a group. Let $g \in G$ be an element in G . Suppose that either the order $|g|$ is infinite or that the order $|g|$ is finite and not invertible in R .

Then for every finitely generated projective RG -module P the value of its Hattori-Stallings rank $\text{HS}_{RG}(P)$ at (g) is trivial.

Sometimes the Bass Conjecture 2.99 for integral domains as coefficients is called the *Strong Bass Conjecture*, see [104, 4.5]. The *Weak Bass Conjecture*, see [104, 4.4], states for a finitely generated projective $\mathbb{Z}G$ -module P that the evaluation of its Hattori-Stallings rank at the unit $\text{HS}_{\mathbb{Z}G}(P)(1)$ agrees with $\dim_{\mathbb{Z}}(\mathbb{Z} \otimes_{\mathbb{Z}G} P)$. Note that $\text{HS}_{\mathbb{Z}G}(P)(1)$ is the same as the von Neumann dimension $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathbb{Z}G} P)$ for a finitely generated projective $\mathbb{Z}G$ -module P , see [650, Corollary 9.61 on page 362].

Exercise 2.100. Show that the Weak Bass Conjecture follows from the Bass Conjecture 2.99 for integral domains as coefficients.

The Bass Conjecture 2.99 can be interpreted topologically. Namely, the Bass Conjecture 2.99 is true for a finitely presented group G in the case $R = \mathbb{Z}$ if and only if every homotopy idempotent self-map of an oriented smooth closed manifold whose dimension is greater than 2 and whose fundamental group is isomorphic to G , is homotopic to one that has precisely one fixed point, see [132]. The Bass Conjecture 2.99 for G in the case $R = \mathbb{Z}$ (or $R = \mathbb{C}$) also implies for a finitely dominated CW -complex with fundamental group G that its Euler characteristic agrees with the L^2 -Euler characteristic of its universal covering, see [327, 0.3].

The next results follows from the argument in [372, Section 5].

Theorem 2.101 (The Farrell-Jones Conjecture and the Bass Conjecture for integral domains). Let G be a group. Suppose that

$$I(G, F) \otimes_{\mathbb{Z}} \mathbb{Q}: \text{colim}_{\text{Or}_{\mathcal{FIN}}(G)} K_0(FH) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective for all fields F of prime characteristic.

Then the Bass Conjecture 2.99 is satisfied for G and every commutative integral domain R .

In particular, the Bass Conjecture 2.99 follows from the Farrell-Jones Conjecture 2.72.

For finite G and R an integral domain such that no prime dividing the order of $|G|$ is a unit in R , Conjecture 2.99 was proved by Swan [937, Theorem 8.1], see also [104, Corollary 4.2]. The Bass Conjecture 2.99 has been proved by Bass [104, Proposition 6.2 and Theorem 6.3] for $R = \mathbb{C}$ and G a torsionfree linear group and by Eckmann [325, Theorem 3.3] for $R = \mathbb{Q}$, provided that G has at most cohomological dimension 2 over $\mathbb{Q}$.

The following result is due to Linnell [632, Lemma 4.1].

Theorem 2.102 (The Bass Conjecture for integral domains and elements of finite order). *Let G be a group.*

- (i) *Let p be a prime, and let P be a finitely generated projective $\mathbb{Z}_{(p)}G$ -module. Suppose for $g \in G$ that $\text{HS}(P)(g) \neq 0$. Then there exists an integer $n \geq 1$ such that g and g^{p^n} are conjugate in G and we get for the Hattori-Stallings rank $\text{HS}(P)(g) = \text{HS}(P)(g^{p^n})$;*
- (ii) *Let P be a finitely generated projective $\mathbb{Z}G$ -module. Suppose for $g \in G$ that $g \neq 1$ and $\text{HS}(P)(g) \neq 0$. Then there exist subgroups C, H of G such that $g \in C$, $C \subseteq H$, C is isomorphic to the additive group $\mathbb{Q}$, H is finitely generated, and the elements of C lie in finitely many H -conjugacy classes. In particular the order of g is infinite.*

More information about the Bass Conjectures can be found in [103, 131, 133, 189, 234, 336, 337, 338, 546, 650, 788, 893, 894].

2.11 The Passage from the Integral to the Rational Group Ring

The following conjecture is taken from [673, Conjecture 85 on page 754].

Conjecture 2.103 (The rational $\tilde{K}_0(\mathbb{Z}G)$ -to- $\tilde{K}_0(\mathbb{Q}G)$ -Conjecture). The change of ring maps

$$\mathbb{Q} \otimes_{\mathbb{Z}} \tilde{K}_0(\mathbb{Z}G) \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \tilde{K}_0(\mathbb{Q}G)$$

is trivial.

If G satisfies the Farrell-Jones Conjecture 2.67 for $K_0(RG)$ for regular R with $\mathbb{Q} \subseteq R$, then it satisfies the rational $\tilde{K}_0(\mathbb{Z}G)$ -to- $\tilde{K}_0(\mathbb{Q}G)$ -Conjecture 2.103, see [673, Proposition 87 on page 754].

Remark 2.104. The question whether an integral version of Conjecture 2.103 holds, i.e., whether the change of ring maps

$$\tilde{K}_0(\mathbb{Z}G) \rightarrow \tilde{K}_0(\mathbb{Q}G)$$

is trivial, is discussed in [673, Remark 89 on page 756].

The answer is no in general. Counterexamples have been constructed by Lehner [625], who actually carefully analyzes the image of the map $\tilde{K}_0(\mathbb{Z}G) \rightarrow \tilde{K}_0(\mathbb{Q}G)$. The group $G = QD_{32} *_{Q_{16}} QD_{32}$ is a counterexample, where QD_{32} is the quasi-dihedral group of order 32, and Q_{16} is the generalized quaternion group of order 16, see [625, Theorem 1.5].

2.12 Survey on Computations of $K_0(RG)$ for Finite Groups

In this section we give a brief survey about computations of $K_0(RG)$ for finite groups G and certain rings R . The upshot will be that the reduced projective class group $\tilde{K}_0(\mathbb{Z}G)$ is a finite abelian group, but in most cases it is non-trivial and unknown, and that for F a field of characteristic zero $K_0(FG)$ is a well-known finitely generated free abelian group.

The following result is due to Swan [937, Theorem 8.1 and Proposition 9.1].

Theorem 2.105 ($\tilde{K}_0(RG)$ is finite for finite G and R the ring of integers in an algebraic number field). *Let G be a finite group. Let R be the ring of algebraic integers in an algebraic number field, e.g., $R = \mathbb{Z}$. Then $\tilde{K}_0(RG)$ is finite.*

A proof of the next theorem will be given in Section 3.8. It was originally proved by Rim [852].

Theorem 2.106 (Rim's Theorem). *Let p be a prime number. The homomorphism induced by the ring homomorphism $\mathbb{Z}[\mathbb{Z}/p] \rightarrow \mathbb{Z}[\exp(2\pi i/p)]$ sending the generator of $\mathbb{Z}/p$ to the primitive p -th root of unity $\exp(2\pi i/p)$*

$$K_0(\mathbb{Z}[\mathbb{Z}/p]) \xrightarrow{\cong} K_0(\mathbb{Z}[\exp(2\pi i/p)])$$

is a bijection.

Example 2.107 ($\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/p])$). Let p be a prime. We have already mentioned in Remark 2.23 that $\mathbb{Z}[\exp(2\pi i/p)]$ is the ring of integers in the algebraic number field $\mathbb{Q}[\exp(2\pi i/p)]$ and hence a Dedekind domain and that the structure of its ideal class group $C(\mathbb{Z}[\exp(2\pi i/p)])$ is only known for a few primes. Thus the message of Rim's Theorem 2.106 is that we know the structure of the finite abelian group $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/p])$ only for a few primes. Here is a table taken from [727, page 30] or [990, Tables §3 on page 352ff].

p	$\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/p])$
≤ 19	$\{0\}$
23	$\mathbb{Z}/3$
29	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$
31	$\mathbb{Z}/9$
37	$\mathbb{Z}/37$
41	$\mathbb{Z}/11 \oplus \mathbb{Z}/11$
43	$\mathbb{Z}/211$
47	$\mathbb{Z}/5 \oplus \mathbb{Z}/139$

Remark 2.108 (Strategy to study $\tilde{K}_0(\mathbb{Z}G)$ for finite G). A $\mathbb{Z}$ -order Λ is a $\mathbb{Z}$ -algebra that is finitely generated projective over $\mathbb{Z}$. Its *locally free class group* is defined as the subgroup of $K_0(\Lambda)$

$$(2.109) \quad Cl(\Lambda) := \{[P] - [Q] \mid P_{(p)} \cong_{\Lambda_{(p)}} Q_{(p)} \text{ for all primes } p\}$$

where (p) denotes localization at the prime p . This is the part of $K_0(\Lambda)$ that can be described by localization sequences. Its significance for $\Lambda = \mathbb{Z}G$ lies in the result of Swan [937], see also Curtis-Reiner [271, Theorem 32.11 on page 676] and [272, (49.12 on page 221)], that $\tilde{K}_0(\mathbb{Z}G) \cong Cl(\mathbb{Z}G)$ for every finite group G . Now fix a maximal $\mathbb{Z}$ -order $\mathbb{Z}G \subseteq \mathcal{M} \subseteq \mathbb{Q}G$. Such a maximal order has better ring properties than $\mathbb{Z}G$, namely, it is a hereditary ring. The map $i_*: Cl(\mathbb{Z}G) \rightarrow Cl(\mathcal{M})$ induced by the inclusion $i: \mathbb{Z}G \rightarrow \mathcal{M}$ is surjective. Define

$$(2.110) \quad D(\mathbb{Z}G) = \ker(i_*: Cl(\mathbb{Z}G) \rightarrow Cl(\mathcal{M})).$$

The definition of $D(\mathbb{Z}G)$ is known to be independent of the choice of the maximal order $\mathcal{M}$. Thus the study of $\tilde{K}_0(\mathbb{Z}G)$ splits into the study of $D(\mathbb{Z}G)$ and $Cl(\mathcal{M})$. The analysis of $Cl(\mathcal{M})$ can be intractable and involves studying cyclotomic fields, whereas the analysis of $D(\mathbb{Z}G)$ essentially uses p -adic logarithms.

Remark 2.111 (Finiteness obstructions and $D(\mathbb{Z}G)$). Often calculations concerning finiteness obstructions are done by first showing that its image in $Cl(\mathcal{M}) = \tilde{K}_0(\mathbb{Z}G)/D(\mathbb{Z}G)$ is trivial, and then determining it in $D(\mathbb{Z}G)$. For instance, Mislin [739] proved that the finiteness obstruction for every finitely dominated homologically nilpotent space with the finite group G as fundamental group lies in $D(\mathbb{Z}G)$, but that not every element in $D(\mathbb{Z}G)$ occurs this way. Questions concerning the Spherical Space Form Problem involve direct computations in $D(\mathbb{Z}G)$, see for instance Bentzen [122], Bentzen-Madsen [123], and Milgram [719]. The group $D(\mathbb{Z}G)$ enters also in the work of Oliver on actions of finite groups on disks, see [771, 772].

For computations of $D(\mathbb{Z}G)$ for finite p -groups we refer to Oliver [773, 774] and Oliver-Taylor [777].

A survey on $D(\mathbb{Z}G)$ and the methods of its computations can be found in Oliver [775].

Theorem 2.112 (Vanishing results for $D(\mathbb{Z}G)$).

- (i) Let G be a finite abelian group G . Then $D(\mathbb{Z}G) = 0$ holds if and only if G satisfies one of the conditions:
 - (a) G has prime order;
 - (b) G is cyclic of order 4, 6, 8, 9, 10, 14;
 - (c) G is $\mathbb{Z}/2 \times \mathbb{Z}/2$;
- (ii) If G is a finite group that is not abelian and satisfies $D(\mathbb{Z}G) = 0$, then it is D_{2n} for $n \geq 3$, A_4 , A_5 , or S_4 ;
- (iii) One has $D(\mathbb{Z}G) = 0$ if G is A_4 , A_5 or S_4 ;
- (iv) $D(\mathbb{Z}D_{2n}) = 0$ for $n < 60$ and $D(\mathbb{Z}D_{120}) = \mathbb{Z}/2$;
- (v) $D(\mathbb{Z}D_{2n}) = 0$ if n satisfies one of the following conditions:
 - (a) n is an odd prime;
 - (b) n is a power of a regular odd prime;
 - (c) n is a power of 2.

Proof. (i) This is proved by Cassou-Nogués [218], see also [272, Theorem 50.16 on page 253].

(ii) This is proved in Endo-Hironaka [339], see also [272, Theorem 50.29 on page 266].

(iii) This follows from Reiner-Ulom [849], see also [272, Theorem 50.29 on page 266].

(iv) This is proved in Endo-Miyata [340], see [272, Theorem 50.30 on page 266].

(v) This is proved in Endo-Hironaka [339], see also [272, Theorem 50.29 on page 266]. $\square$

Theorem 2.113 (Finite groups with vanishing $\widetilde{K}_0(\mathbb{Z}G)$).

(i) Let G be a finite abelian group G . Then $\widetilde{K}_0(\mathbb{Z}G) = 0$ holds if and only if G satisfies one of the conditions:

- (a) G is cyclic of order n for $1 \leq n \leq 11$;
- (b) G is cyclic of order 13, 14, 17, 19;
- (c) G is $\mathbb{Z}/2 \times \mathbb{Z}/2$;

(ii) If G is a non-abelian finite group with $\widetilde{K}_0(\mathbb{Z}G) = 0$, then G is D_{2n} for $n \geq 3$, A_4 , A_5 , or S_4 ;

(iii) We have $\widetilde{K}_0(\mathbb{Z}G) = 0$ for $G = A_4, S_4, D_6, D_8, D_{12}$.

Proof. (i) This is proved by Cassou-Nogués [218], see also [272, Corollary 50.17 on page 253].

(ii) This follows from Theorem 2.112 (ii).

(iii) The cases $G = A_4, S_4, D_6, D_8$ are already treated in [848, Theorem 6.4 and Theorem 8.2]. Because of Theorem 2.112 (iii) it suffices to show for the maximal order $\mathcal{M}$ for the groups $G = A_4, S_4, D_6, D_8, D_{12}$ that $Cl(\mathcal{M}) = 0$. This follows from the fact that $\mathbb{Q}G$ is a products of matrix algebras over $\mathbb{Q}$ and hence the maximal $\mathbb{Z}$ -order $\mathcal{M}$ is a products of matrix rings over $\mathbb{Z}$. $\square$

Exercise 2.114. Determine all finite groups G of order ≤ 9 for which $\widetilde{K}_0(\mathbb{Z}G)$ is non-trivial.

Theorem 2.115 ($K_0(RG)$ for finite G and an Artinian ring R). Let R be an Artinian ring. Let G be a finite group. Then RG is also an Artinian ring. There are only finitely many isomorphism classes $[P_1], [P_2], \dots, [P_n]$ of irreducible finitely generated projective RG -modules, and we obtain an isomorphism of abelian groups

$$\mathbb{Z}^n \xrightarrow{\cong} K_0(RG), \quad (k_1, k_2, \dots, k_n) \mapsto \sum_{i=1}^n k_i \cdot [P_i].$$

Proof. This follows from [271, Proposition 16.7 on page 406 and the paragraph after Corollary 6.22 on page 132]. $\square$

Let F be a field of characteristic zero or of characteristic p for a prime number p not dividing $|G|$. Then $K_0(FG)$ is the same as the representation ring $\text{Rep}_F(G)$ of G with coefficients in the field F since the ring FG is *semisimple* i.e., every submodule of a module is a direct summand. If F is a field of characteristic zero, then representations are detected by their characters, see Lemma 2.89. For more information about modules over FG for a finite group G and a field F we refer for instance to Curtis-Reiner [271, Chapter 1 and Chapter 2] and Serre [908].

Exercise 2.116. Compute $K_0(FD_8)$ for $F = \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$.

2.13 Survey on Computations of $K_0(C_r^*(G))$ and $K_0(\mathcal{N}(G))$

Let G be a group. Let $\mathcal{B}(L^2(G))$ denote the algebra of bounded linear operators on the Hilbert space $L^2(G)$ whose orthonormal basis is G . The *reduced group C^* -algebra* $C_r^*(G)$ is the closure in the norm topology of the image of the regular representation $\mathbb{C}G \rightarrow \mathcal{B}(L^2(G))$ that sends an element $u \in \mathbb{C}G$ to the (left) G -equivariant bounded operator $L^2(G) \rightarrow L^2(G)$ given by right multiplication with u^{-1} . The *group von Neumann algebra* $\mathcal{N}(G)$ is the closure in the weak topology. There is an identification $\mathcal{N}(G) = \mathcal{B}(L^2(G))^G$. One has natural inclusions

$$\mathbb{C}G \subseteq C_r^*(G) \subseteq \mathcal{N}(G) \subseteq \mathcal{B}(L^2(G)).$$

We have $\mathbb{C}G = C_r^*(G) = \mathcal{N}(G)$ if and only if G is finite. If $G = \mathbb{Z}$, then the Fourier transform gives identifications $C_r^*(\mathbb{Z}) = C(S^1)$ and $\mathcal{N}(\mathbb{Z}) = L^\infty(S^1)$.

Remark 2.117 ($K_0(C_r^*(G))$ versus $K_0(\mathbb{C}G)$). We will later see that the study of $K_0(C_r^*(G))$ is not done according to its algebraic nature. Instead we will introduce and analyze the topological K -theory of $C_r^*(G)$ and explain that in dimension 0 the algebraic and the topological K -theory of $C_r^*(G)$ agree. In order to explain the different flavor of $K_0(C_r^*(G))$ in comparison with $K_0(\mathbb{C}G)$, we mention the conclusion of the Baum-Connes Conjecture for torsionfree groups 10.44 that for torsionfree G there exists an isomorphism

$$\bigoplus_{n \geq 0} H_{2n}(BG; \mathbb{Q}) \xrightarrow{\cong} K_0(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The space BG is the *classifying space of the group G* , which is up to homotopy characterized by the property that it is a CW -complex with $\pi_1(BG) \cong G$ whose universal covering is contractible. We denote by $H_*(X, R)$ the *singular or cellular homology* of a space or CW -complex X with coefficient in a commutative ring R . We can identify $H_*(BG; R)$ with the group homology of G with coefficients in R .

We see that $K_0(C_r^*(G))$ can be huge also for torsionfree groups, whereas $K_0(\mathbb{C}G) \cong \mathbb{Z}$ for torsionfree G is a conclusion of the Farrell-Jones Conjecture 2.60 for $K_0(R)$ for torsionfree G and regular R . We see already here a homological be-

havior of $K_0(C_r^*(G))$, which is not yet evident in the case of group rings so far and will become clear later.

Remark 2.118 ($K_0(\mathcal{N}(G))$). The projective class group $K_0(\mathcal{A})$ can be computed for any von Neumann algebra $\mathcal{A}$ using the center-valued universal trace, see for instance [650, Section 9.2]. In particular one gets for a finitely generated group G that does not contain $\mathbb{Z}^n$ as subgroup of finite index an isomorphism

$$K_0(\mathcal{N}(G)) \cong \mathcal{Z}(\mathcal{N}(G))^{\mathbb{Z}/2}.$$

Here $\mathcal{Z}(\mathcal{N}(G))$ is the center of the group von Neumann algebra and the $\mathbb{Z}/2$ -action comes from taking the adjoint of an operator in $\mathcal{B}(L^2(G))$, see [650, Example 9.34 on page 353]. If G is a finitely generated group that does not contain $\mathbb{Z}^n$ as subgroup of finite index and for which the conjugacy class (g) of an element g different from the unit is always infinite, then $\mathcal{Z}(\mathcal{N}(G)) = \mathbb{C}$ and one obtains an isomorphism

$$K_0(\mathcal{N}(G)) \cong \mathbb{R}.$$

A pleasant feature of $\mathcal{N}(G)$ is that there is no difference between stably isomorphic and isomorphic in the sense that for three finitely generated projective $\mathcal{N}(G)$ -modules P_0, P_1 , and Q we have $P_0 \oplus Q \cong_{\mathcal{N}(G)} P_1 \oplus Q$ if and only if $P_0 \cong_{\mathcal{N}(G)} P_1$.

We see that in the case of the group von Neumann algebra we can compute $K_0(\mathcal{N}(G))$ completely, but the answer does not show any homological behavior in G . In fact, the Farrell-Jones Conjecture and the Baum-Connes Conjecture have no analog for group von Neumann algebras.

Exercise 2.119. Let G be a torsionfree hyperbolic group that is not cyclic. Prove $K_0(\mathcal{N}(G)) \cong \mathbb{R}$.

Remark 2.120 (Change of rings homomorphisms for $\tilde{K}_0$ for $\mathbb{Z}G \rightarrow \mathbb{C}G \rightarrow C_r^*(G) \rightarrow \mathcal{N}(G)$). We summarize what is conjectured or known about the string of change of rings homomorphism

$$\tilde{K}_0(\mathbb{Z}G) \xrightarrow{i_1} \tilde{K}_0(\mathbb{C}G) \xrightarrow{i_2} \tilde{K}_0(C_r^*(G)) \xrightarrow{i_3} \tilde{K}_0(\mathcal{N}(G))$$

coming from the various inclusion of rings. The first map i_1 is conjectured to be rationally trivial, see [673, Conjecture 85 on page 754], but is not integrally trivial, see [625, Theorem 5.1]. The second map i_2 is conjectured to be rationally injective, compare [649, Theorem 0.5], but is not surjective in general. The map i_3 is in general not injective, not surjective, and not trivial. It is known that the composite $i_3 \circ i_2 \circ i_1$ is trivial, see for instance [650, Theorem 9.62 on page 362].

2.14 Notes

Algebraic K -theory is compatible with direct limits, as explained for the projective class group next. A *directed set* I is a non-empty set with a partial ordering $\leq$ such that for two elements i_0 and i_1 there exists an element i with $i_0 \leq i$ and $i_1 \leq i$. A *directed system of rings* is a set of rings $\{R_i \mid i \in I\}$ indexed by a directed set I together with a choice of a ring homomorphism $\phi_{i,j}: R_i \rightarrow R_j$ for $i, j \in I$ with $i \leq j$ such that $\phi_{i,k} = \phi_{j,k} \circ \phi_{i,j}$ holds for $i, j, k \in I$ with $i \leq j \leq k$ and $\phi_{i,i} = \text{id}$ holds for $i \in I$. The *colimit*, sometimes also called the *direct limit*, of $\{R_i \mid i \in I\}$ is a ring denoted by $\text{colim}_{i \in I} R_i$ together with ring homomorphisms $\psi_j: R_j \rightarrow \text{colim}_{i \in I} R_i$ for every $j \in I$ such that $\psi_j \circ \phi_{i,j} = \psi_i$ holds for $i, j \in I$ with $i \leq j$ and the following universal property is satisfied: For every ring S and every system of ring homomorphisms $\{\mu_i: R_i \rightarrow S \mid i \in I\}$ such that $\mu_j \circ \phi_{i,j} = \mu_i$ holds for $i, j \in I$ with $i \leq j$, there is precisely one ring homomorphism $\mu: \text{colim}_{i \in I} R_i \rightarrow S$ satisfying $\mu \circ \psi_i = \mu_i$ for every $i \in I$. If we replace ring by group or module everywhere, we get the notion of directed system and direct limit of groups or modules respectively. This is a special case of the direct limit of a functor, namely, consider I as category with the set I as objects and precisely one morphism from i to j if $i \leq j$, and no other morphisms.

Remark 2.121 (Filtered categories). One may consider instead of a directed set a *filtered category*, i.e., a nonempty category I such that for every two objects i and j there is an object k together with two morphisms $i \rightarrow k$ and $j \rightarrow k$ and for two morphism $f, g: i \rightarrow j$ with the same source and target there is a morphism $h: j \rightarrow k$ with $hj \circ f = h \circ k$, and all the results about colimits over directed sets stay true if one considers colimits over filtered categories. Then one talks about filtered systems instead of filtered sets.

Let $\{R_i \mid i \in I\}$ be a direct system of rings. For every $i \in I$, we obtain a change of rings homomorphism $(\psi_i)_*: K_0(R_i) \rightarrow K_0(R)$. The universal property of the direct limit yields a homomorphism

$$(2.122) \quad \text{colim}_{i \in I} (\psi_i)_*: \text{colim}_{i \in I} K_0(R_i) \xrightarrow{\cong} K_0(R),$$

which turns out to be an isomorphism, see [860, Theorem 1.2.5].

We denote by $R^\times$ the group of units in R . A ring R is called *local* if the set $I := R - R^\times$ forms a (left) ideal. If I is a left ideal, it is automatically a two-sided ideal and it is maximal both as a left ideal and as a right ideal. A ring R is local if and only if it has a unique maximal left ideal and a unique maximal right ideal and these two coincide. An example of a local ring is the ring of formal power series $F[[t]]$ with coefficients in a field F . If R is a commutative ring and I is a prime ideal, then the localization R_I of R at I is a local ring.

Theorem 2.123 ($K_0(R)$ of local rings). *Let R be a local ring. Then every finitely generated projective R -module is free and $K_0(R)$ is infinite cyclic with $[R]$ as generator.*

Proof. See for instance [727, Lemma 1.2 on page 5] or [860, Theorem 1.3.11 on page 14]. $\square$

The proof is based on *Nakayama's Lemma*, which says for a ring R and a finitely generated R -module M that $\text{rad}(R)M = M \iff M = 0$ holds. Here $\text{rad}(R)$ is the *radical*, or *Jacobson radical*, i.e., the two-sided ideal that is given by the intersection of all maximal left ideals, or, equivalently, of all maximal right ideals of R . The radical is the same as the set of elements $r \in R$ for which there exists an $s \in S$ such that $1 - rs$ has a left inverse in R .

If R is a commutative ring and $\text{spec}(R)$ is its *spectrum* consisting of its prime ideals and equipped with the Zariski topology, then we obtain for every finitely generated projective R -module P a continuous rank function $\text{Spec}(R) \rightarrow \mathbb{Z}$ by sending a prime ideal I to the rank of the finitely generated free R_I -module $P_I = P \otimes_R R_I$. This makes sense because of Theorem 2.123 since R_I is local. If R is a commutative integral domain, this rank function is constant. For more details we refer for instance to [860, Proposition 1.3.12 on page 15].

Exercise 2.124. Prove for an integer $n \geq 1$ that $K_0(\mathbb{Z}/n)$ is the free abelian group whose rank is the number of prime numbers dividing n .

A ring is called *semilocal* if $R/\text{rad}(R)$ is Artinian, or, equivalently, $R/\text{rad}(R)$ is semisimple. If R is commutative, then R is semilocal if and only if it has only finitely many maximal ideals, see [916, page 69]. For a semilocal ring R , the projective class group $K_0(R)$ is a finitely generated free abelian group, see [916, Proposition 14 on page 28]. More information about semilocal rings can be found for instance in [610, § 20].

Lemma 2.125. For any ring R and nilpotent two-sided ideal $I \subseteq R$, the map $K_0(R) \rightarrow K_0(R/I)$ induced by the projection $R \rightarrow R/I$ is bijective.

Proof. See [998, Lemma 2.2 in Section II.2 on page 70]. $\square$

Given two groups G_1 and G_2 , let $G_1 * G_2$ be the amalgamated free product. Then the natural maps $G_k \rightarrow G_0 * G_1$ for $k = 1, 2$ induce an isomorphism, see [421, Theorem 1.1],

$$(2.126) \quad \tilde{K}_0(\mathbb{Z}[G_1]) \oplus \tilde{K}_0(\mathbb{Z}[G_2]) \cong \tilde{K}_0(\mathbb{Z}[G_1 * G_2]).$$

This is a first glimpse of a homological behavior of K_0 if one compares this with the corresponding isomorphism of group homology

$$\tilde{H}_n(G_1) \oplus \tilde{H}_n(G_2) \cong \tilde{H}_n(G_1 * G_2).$$

Exercise 2.127. Show that the projections $\text{pr}_k: G_1 \times G_2 \rightarrow G_k$ for $k = 1, 2$ do not in general induce isomorphisms

$$\tilde{K}_0(\mathbb{Z}[G_1 \times G_2]) \rightarrow \tilde{K}_0(\mathbb{Z}[G_1]) \times \tilde{K}_0(\mathbb{Z}[G_2]).$$

There are also equivariant versions of the finiteness obstructions, see for instance [32], [642], and [644, Chapter 3 and 11]. Finiteness obstructions for categories are investigated in [391, 390].

Andrej Jaikin-Zapirain pointed out that he and Pablo Sánchez-Peralta have proved the following result confirming Conjecture 2.60 in a special case.

A presentation $G = \langle X \mid R \rangle$ is called a Cohen–Lyndon presentation if for each $r \in R$, there exists a transversal T_r of the normal subgroup $N = \langle\langle R \rangle\rangle$, such that N is freely generated by the set $\{r^g \mid r \in R, g \in T_r\}$.

They prove that if G has a Cohen–Lyndon presentation and S is a regular ring, then the natural map

$$K_0(S) \rightarrow K_0(S[G])$$

is an isomorphism.